

## HW 7

Due on March 23, 2020 (in class).

### Orthogonal and unitary matrices.

1. Let  $V$  be an inner product space over  $\mathbf{C}$ . Show that a linear transformation  $T: V \rightarrow V$  is unitary if and only if  $T$  preserves norms.
2. Let  $U_n$  be the set of unitary  $n \times n$  matrices. Show that  $U_n$  is a compact subset of  $M_n(\mathbf{C})$ . [**Hint:** Consider the map  $A \mapsto \bar{A}^t A$  on  $M_n(\mathbf{C})$ .]

**Implicit function theorem.** Let  $n = d + m$  where  $d$  and  $m$  are non-negative integers. Let  $S$  be a subset of  $\{1, \dots, n\}$  of cardinality  $m$ , say  $S = \{j_1, \dots, j_m\}$  with  $j_1 < \dots < j_m$ . Let  $\{i_1, \dots, i_d\}$  be its complement in  $\{1, \dots, n\}$  with  $i_1 < \dots < i_d$ . We then have an automorphism of  $\mathbf{R}^n$

$$\Theta_S: \mathbf{R}^n \longrightarrow \mathbf{R}^n$$

with  $\Theta_S(z_1, \dots, z_n) = (z_{i_1}, \dots, z_{i_d}, z_{j_1}, \dots, z_{j_m})$ . For simplicity, we will continue to use the symbol  $\Theta_S$  for  $\Theta_S|_U$  for any subset  $U$  of  $\mathbf{R}^n$ . More often than not  $U$  will be open in  $\mathbf{R}^n$ .

As usual, let  $\pi: \mathbf{R}^n \rightarrow \mathbf{R}^d$  be the projection to the first  $d$  coordinates and  $\pi': \mathbf{R}^n \rightarrow \mathbf{R}^m$  the projection to the last  $m$  coordinates.

Fix an open subset  $U$  of  $\mathbf{R}^n$ , and let  $U_S = \Theta_S(U)$ . Let  $\varphi: U \rightarrow \mathbf{R}^m$  be  $\mathcal{C}^1$  and let  $C_1, \dots, C_n$  be the columns of  $J\varphi$ . Fix  $\mathbf{p} = (p_1, \dots, p_n) \in U$ , and let  $\mathbf{c} = \varphi(\mathbf{p})$ , and  $M = \varphi^{-1}(\mathbf{c})$ . Let  $S, j_1, \dots, j_m, i_1, \dots, i_d$  be as above. Set  $\mathbf{a}_S = (p_{i_1}, \dots, p_{i_d}) \in \mathbf{R}^d$ , and  $\mathbf{b}_S = (p_{j_1}, \dots, p_{j_m}) \in \mathbf{R}^m$ .

3. Suppose  $C_{j_1}(\mathbf{p}), \dots, C_{j_m}(\mathbf{p}) \in \mathbf{R}^m$  are linearly independent. Show that there is an open subset  $W_S$  of  $\mathbf{R}^d$  containing  $\mathbf{a}_S$  and a unique  $\mathcal{C}^1$  map  $\mathbf{g}_S: W_S \rightarrow \mathbf{R}^n$  taking values in  $M^1$  such that  $\mathbf{g}_S(\mathbf{a}_S) = \mathbf{p}$  and such that  $\pi \circ \Theta_S \circ \mathbf{g}_S$  is the identity map on  $W_S$ . [**Hint:** Examine the map  $\varphi_S: U_S \rightarrow \mathbf{R}^m$  given by  $\varphi_S = \varphi \circ \Theta_S^{-1}$ .]
4. Let  $C_{j_1}(\mathbf{p}), \dots, C_{j_m}(\mathbf{p})$  be linearly independent and  $\mathbf{g}_S$  be as in Problem 3. Show that  $\mathbf{g}_S$  is one-to-one and takes open subsets of  $W_S$  to open subsets of  $M$ . [**Hint:** Re-examine how the implicit function theorem follows from the inverse function theorem.]
5. Let  $C_{j_1}(\mathbf{p}), \dots, C_{j_m}(\mathbf{p})$  be linearly independent. Let  $T$  be another subset of  $\{1, \dots, n\}$  of cardinality  $m$  and suppose the set  $\{C_j(\mathbf{p}) \mid j \in T\}$  is also linearly independent. Let  $\mathbf{g}_S$  and  $\mathbf{g}_T$  be as in Problem 3. Let  $V$  be the open subset of  $M$  given by  $V = \mathbf{g}_S(W_S) \cap \mathbf{g}_T(W_T)$  and let  $V_S$  and  $V_T$  be the open subsets of  $\mathbf{R}^d$  given by  $V_S = \mathbf{g}_S^{-1}(V)$  and  $V_T = \mathbf{g}_T^{-1}(V)$ . Show that the map

$$(\mathbf{g}_T|_{V_T})^{-1} \circ (\mathbf{g}_S|_{V_S}): V_S \longrightarrow V_T$$

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<sup>1</sup>This means  $\mathbf{g}_S(\mathbf{x}) \in M$  for every  $\mathbf{x} \in W_S$ .

is  $\mathcal{C}^1$ . [**Advice:** It may be notationally simpler to assume (and this can be done without loss of generality) that  $T = \{d + 1, \dots, n\}$ .]

6. Read the definition of a differential manifold from any book. Relate the above to the notion of a differentiable manifold when  $\varphi'(z)$  has rank  $m$  at every  $z \in M$  (assume  $\varphi$  is  $\mathcal{C}^\infty$  if the definition you read needs  $\mathcal{C}^\infty$  functions). You don't have to submit this problem.
7. Show that  $U_n$  is a differentiable manifold. Again you don't have to submit this problem.
8. Take a five kilometre walk every evening. This one has to be submitted.