

## HW 6

Due on March 16, 2020 (in class).

**Basic linear algebra.** Suppose  $V$  and  $W$  are finite dimensional vector spaces over a field  $k$ , with  $v_1, \dots, v_n$  a basis for  $V$  and  $w_1, \dots, w_m$  a basis for  $W$ . Let  $R: V \xrightarrow{\sim} k^n$  and  $L: W \xrightarrow{\sim} k^m$  be the isomorphisms given by  $Rv_i = \mathbf{e}_i$ ,  $i = 1, \dots, n$  and  $Lw_j = \mathbf{e}'_j$ ,  $j = 1, \dots, m$ , where  $\mathbf{e}_1, \dots, \mathbf{e}_n$  and  $\mathbf{e}'_1, \dots, \mathbf{e}'_m$  are the standard bases on  $k^n$  and  $k^m$  respectively. Let  $T: V \rightarrow W$  be a linear transformation and  $S: k^n \rightarrow k^m$  the linear transformation defined by the commutativity of the diagram:

$$\begin{array}{ccc}
 W & \xleftarrow{T} & V \\
 L \downarrow \} & & \} \downarrow R \\
 k^m & \xleftarrow{S} & k^n
 \end{array}$$

In other words,  $S = L \circ T \circ R^{-1}$ . ( $R$  is for “right” and  $L$  is for “left” and note that  $LT = SR$ .)

If a basis for a vector space is listed as  $b_1, \dots, b_r$ , we assume it is an ordered basis, with the subscripts giving the order. Thus in the above, the matrices with respect to the various bases make sense, since we treat them as ordered bases.

1. Show that the matrix of  $T$  with respect to the bases  $v_1, \dots, v_n$  and  $w_1, \dots, w_m$  is the same as the matrix of  $S$  with respect to the standard ordered bases on  $k^n$  and  $k^m$ .

**Change of coordinates.** Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  and  $\mathbf{e}'_1, \dots, \mathbf{e}'_m$  be the standard bases on  $\mathbf{R}^n$  and  $\mathbf{R}^m$  respectively. Let  $\mathbf{v}_1, \dots, \mathbf{v}_n$  be a basis for  $\mathbf{R}^n$  and  $\mathbf{w}_1, \dots, \mathbf{w}_m$  a basis for  $\mathbf{R}^m$ ,  $R$  the linear automorphism of  $\mathbf{R}^n$  which sends  $\mathbf{v}_i$  to  $\mathbf{e}_i$  for every  $i$ ,  $L$  the linear automorphism of  $\mathbf{R}^m$  which sends  $\mathbf{w}_j$  to  $\mathbf{e}'_j$  for every  $j$ . Let  $U$  be an open subset of  $\mathbf{R}^n$ ,  $\mathbf{f}: U \rightarrow \mathbf{R}^m$  a map, and  $U' = R(U)$ .

2. Show that  $U'$  is open in  $\mathbf{R}^n$ .
3. Fix  $j \in \{1, \dots, m\}$ . For  $x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n \in U$ , let  $g_j(x_1, \dots, x_n)$  be the coefficient of  $\mathbf{w}_j$  when we write  $\mathbf{f}(x_1\mathbf{v}_1 + \dots + x_n\mathbf{v}_n)$  as a linear combination of  $\mathbf{w}_1, \dots, \mathbf{w}_m$ . For  $(x_1, \dots, x_n) \in U'$  show that  $g_j(x_1, \dots, x_n)$  makes sense and that the assignment  $(x_1, \dots, x_n) \mapsto g_j(x_1, \dots, x_n)$  gives a map  $g_j: U' \rightarrow \mathbf{R}$ .

4. Let  $\mathbf{g} = (g_1, \dots, g_m)$  where the  $g_j$  are as in Problem 3. Show that the following diagram commutes

$$\begin{array}{ccc} \mathbf{R}^m & \xleftarrow{f} & U \\ L \downarrow & & \downarrow R|_U \\ \mathbf{R}^m & \xleftarrow{g} & U' \end{array}$$

5. Let  $\mathbf{p} \in U$ , and  $\mathbf{q} = R(\mathbf{p})$ . Let  $\mathbf{g}$  be as in Problem 4. Suppose  $\mathbf{f}$  is differentiable at  $\mathbf{p}$ .
- Show that  $\mathbf{g}$  is differentiable at  $\mathbf{q}$ .
  - Show that the matrix of  $\mathbf{f}'(\mathbf{p})$  with respect to the bases  $\mathbf{v}_1, \dots, \mathbf{v}_n$  and  $\mathbf{w}_1, \dots, \mathbf{w}_m$  is  $(J\mathbf{g})(\mathbf{q})$ . [**Hint:** “Differentiate” the diagram in Problem 4. and use Problem 1.]

**Maxima and minima.** Let  $U$  be an open subset of  $\mathbf{R}^n$  and  $h: U \rightarrow \mathbf{R}$ ,  $f: U \rightarrow \mathbf{R}$  two  $\mathcal{C}^1$  functions on  $U$ . Let  $c \in h(U)$  and let  $M$  be the level set  $M = \{\mathbf{x} \in U \mid h(\mathbf{x}) = c\}$ . We say  $f|_M$  has a *local maximum* at  $\mathbf{v} \in M$  if there exists an open neighbourhood  $W$  of  $\mathbf{v}$  in  $U$  such that  $f(\mathbf{x}) \leq f(\mathbf{v})$ , for all  $\mathbf{x} \in W \cap M$ . It is said to have a *local minimum* at  $\mathbf{v}$  if there exists an open neighbourhood  $W$  of  $\mathbf{v}$  such that  $f(\mathbf{x}) \geq f(\mathbf{v})$  for all  $\mathbf{x} \in W \cap M$ . Finally,  $f|_M$  is said to have a *local extremum* at  $\mathbf{v}$  if it has either a local maximum or a local minimum at  $\mathbf{v}$ .

6. Suppose  $h'$  does not vanish at any point of  $M$ .
- Show that  $M$  has a tangent space at each of its points. (See [Homework 5](#) for the definition. You will also need to look at the definition given in [Homework 4](#) which was modified in Homework 5.)
  - Suppose  $f|_M$  has a local extremum at  $\mathbf{v} = (v_1, \dots, v_m) \in M$ . Show that  $\nabla f(\mathbf{v})$  is orthogonal to the tangent space of  $M$  at  $\mathbf{v}$ . [**Hint:** Consider  $\mathcal{C}^1$  paths through  $\mathbf{v}$  and take velocity vectors of these paths. Next use the chain rule and one variable calculus.]
7. Suppose  $h'$  does not vanish at any point of  $M$  and suppose  $f|_M$  has a local extremum at  $\mathbf{v} \in M$ . Show that there is a unique scalar  $\lambda \in \mathbf{R}$  such that  $\nabla f(\mathbf{v}) = \lambda \nabla h(\mathbf{v})$ .