HW 5

Due on March 11, 2020 (in class).

Inner products. Let $(V, \langle\ ,\ \rangle)$ be a finite dimensional inner product space over $\mathbf{K}$, where $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$. Let H be a subspace of V. Define the *orthogonal complement* $H^\perp$ of H as $H^\perp$ as

$$H^\perp = \{v \in V \mid \langle v, h\rangle = 0 \text{ for all } h \in H\}.$$

A mapping $f\colon X \to Z$ between $\mathbf{K}$ vector spaces is said to be *anti linear* if $f(ax + by) = \bar{a}f(x) + \bar{b}f(y)$ for $a, b \in K$ and $x, y \in X$. Note that if $\mathbf{K} = \mathbf{R}$, then anti linear maps are the same as linear maps. An *anti linear* isomorphism is an anti linear map which is an isomorphism on the underlying abelian groups.

If $V = \mathbf{K}^n$, then by the *usual inner product*, or the *inner product* on V we mean the inner product given by $\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle = \sum_{i=1}^n x_i\bar{y}_i$ where $\bar{y}_i$ is the complex conjugate of y_i. The default inner product on $\mathbf{K}^n$, one we may often not specify, is the usual inner product.

1. Show the following:
  (a) $H^\perp$ is a subspace of V.
  (b) $H^\perp = 0$ if and only if $H = V$.
  (c) $H^{\perp\perp} = H$.

2. Show that $V = H \oplus H^\perp$, i.e. show that every $v \in V$ has a unique decomposition $v = v' + v''$ such that $v' \in H$ and $v'' \in H^\perp$. [**Hint:** Use Gram-Schmidt orthogonalisation, namely that every finite dimensional inner product space W has a basis $w_1, \ldots, w_k$ such that $\langle w_i, w_j\rangle = \delta_{ij}$. Such bases are called *orthonormal* bases, and their existence will be proved in class in one of the upcoming lectures.]

3. For $v \in V$, show that $f_v\colon V \to \mathbf{K}$ given by $f_v(w) = \langle w, v\rangle$ is a linear functional. Let V^* denote the dual of V. Show also that the map $D\colon V \to V^*$ given by $v \mapsto f_v$ is an anti linear isomorphism.

4. Let $T\colon V \to V$ be a $\mathbf{K}$ linear map. Show that there is a unique linear map $T^*\colon V \to V$ such that $\langle Tv, w\rangle = \langle v, T^*w\rangle$ for $v, w \in V$.

Calculus. Let d and m be non-negative integers and $n = d + m$. Identify $\mathbf{R}^n$ with $\mathbf{R}^d \times \mathbf{R}^m$ in the usual way. Suppose $\boldsymbol{\varphi}\colon U \to \mathbf{R}^m$ is a $\mathscr{C}^1$ map where U is an open subset of $\mathbf{R}^n$. Let $\boldsymbol{p} = (\boldsymbol{a}, \boldsymbol{b}) \in U$ be a point such that the rank of $\boldsymbol{\varphi}'(\boldsymbol{p})$ is m. Let $\boldsymbol{c} = \boldsymbol{\varphi}(\boldsymbol{p})$ and M the subset of U consisting points $(\boldsymbol{x}, \boldsymbol{y}) \in U$ such that $\boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{c}$. There are some changes we wish to make regarding terminology and notations used in Homework 4. Here is the list.

- What was called the tangent space of M at $\boldsymbol{p}$ will be called the *geometric* or *classical tangent space of M at $\boldsymbol{p}$*. The geometric tangent space will be denoted $T_{\boldsymbol{p}}^{\mathrm{cl}}$ and not $T_{\boldsymbol{p}}$. (We will use $T_{\boldsymbol{p}}$ for $V_{\boldsymbol{p}}$ instead.)
- The null space of $\boldsymbol{\varphi}'(\boldsymbol{p})$ will be called the *tangent space of M at $\boldsymbol{p}$*. This space was denoted $V_{\boldsymbol{p}}$ in Homework 4. We will discard that in favour of $T_{\boldsymbol{p}}$. So from now on $T_{\boldsymbol{p}}$ is the old $V_{\boldsymbol{p}}$ (and the old $T_{\boldsymbol{p}}$ is now $T_{\boldsymbol{p}}^{\mathrm{cl}}$). Note that $T_{\boldsymbol{p}}^{\mathrm{cl}} = T_{\boldsymbol{p}} + \boldsymbol{p}$.

Using the standard inner product on $\mathbf{R}^n$, the *normal space to M at $\boldsymbol{p}$* is defined to be the subspace $N_{\boldsymbol{p}}$ of $\mathbf{R}^n$ defined by $N_{\boldsymbol{p}} := T_{\boldsymbol{p}}^{\perp}$. The *geometric* or *classical normal space to M at $\boldsymbol{p}$* is $N_{\boldsymbol{p}}^{\mathrm{cl}} := N_{\boldsymbol{p}} + \boldsymbol{p}$.

Finally, as we did earlier, let $\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}}$ and $\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{y}}$ denote that submatrices formed by the first d columns and the last m columns respectively of the Jacobian matrix $J\boldsymbol{\varphi}$. Note that $\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{y}}$ is an $m \times m$ matrix.

5. Show that the columns of $\left((J\boldsymbol{\varphi})(\boldsymbol{p})\right)^t$ form a basis of $N_{\boldsymbol{p}}$. Here $\left((J\boldsymbol{\varphi})(\boldsymbol{p})\right)^t$ is the transpose of the Jacobian matrix of $\boldsymbol{\varphi}$ at $\boldsymbol{p}$.

6. Let $\pi\colon \mathbf{R}^n \to \mathbf{R}^d$ and $\pi'\colon \mathbf{R}^n \to \mathbf{R}^m$ be the usual projections, i.e. $\pi(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}$ and $\pi'(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}$.

(a) Show that π' maps $N_{\boldsymbol{p}}$ surjectively on to $\mathbf{R}^m$ if and only if $\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{y}}(\boldsymbol{p})$ is an invertible $m \times m$ matrix.

(b) Give an if and only if criterion for $\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{y}}(\boldsymbol{p})$ to be invertible in terms of $T_{\boldsymbol{p}}$ and the map π and prove your assertion.

7. Suppose $\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{y}}(\boldsymbol{p})$ is invertible and $\boldsymbol{f}\colon W \to \mathbf{R}^m$ an implicit solution to $\boldsymbol{\varphi}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{c}$ in an open neighbourhood W of $\boldsymbol{a}$ in $\mathbf{R}^d$, with $\boldsymbol{f}(\boldsymbol{a}) = \boldsymbol{b}$. Show that

$$(J\boldsymbol{f})(\boldsymbol{a}) = -\left(\frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{y}}(\boldsymbol{a}, \boldsymbol{b})\right)^{-1} \frac{\partial \boldsymbol{\varphi}}{\partial \boldsymbol{x}}(\boldsymbol{a}, \boldsymbol{b}).$$

Picture: Here is a picture of a curve $\varphi(x, y) = 0$ in the plane, with $\varphi \in \mathscr{C}^1$, together with $T_{\boldsymbol{p}}^{\mathrm{cl}}$ and $N_{\boldsymbol{p}}^{\mathrm{cl}}$ at a point $\boldsymbol{p}$. Note that $N_{\boldsymbol{p}}$ projects surjectively (and hence isomorphically) on to the y-axis via π' and $T_{\boldsymbol{p}}$ projects surjectively (and hence isomorphically) on to the x-axis via π. This is a way of saying, via Problem **6.**, that the hypotheses of the implicit function theorem are true at $\boldsymbol{p}$.