HW 4

Due on Feb 19, 2020 (in class).

The transpose of a matrix A will be written A^t. The standard inner product in $\mathbf{R}^n$ will be denoted $\langle\ ,\ \rangle$.

Velocity vector of a path and the gradient of a function. Let U be an open subset of $\mathbf{R}^n$. Let $f\colon U \to \mathbf{R}$ be a $\mathscr{C}^1$ function. The *gradient of* f is the map

$$\boldsymbol{\nabla} f\colon U \to \mathbf{R}^n$$

given by

$$\boldsymbol{\nabla} f(\boldsymbol{x}) = (D_1 f(\boldsymbol{x}), \dots, D_n f(\boldsymbol{x})) = f'(\boldsymbol{x})^t.$$

If $\boldsymbol{\gamma}\colon (a,b) \to U$ is a smooth path, i.e. $\boldsymbol{\gamma}$ is a $\mathscr{C}^1$ map, then the *velocity of* $\boldsymbol{\gamma}$ at a point $\theta \in (a,b)$ is

$$\boldsymbol{v}_{\boldsymbol{\gamma}}(\theta) = \boldsymbol{\gamma}'(\theta) \in \mathbf{R}^n.$$

If $\boldsymbol{p} = \boldsymbol{\gamma}(\theta)$, we often write $\boldsymbol{v}_{\boldsymbol{\gamma}}(\boldsymbol{p})$ for $\boldsymbol{v}_{\boldsymbol{\gamma}}(\theta)$. The default assumption when this notation is used is that $\boldsymbol{\gamma}$ does not "visit" $\boldsymbol{p}$ more than once (easily achieved by restricting the domain of $\boldsymbol{\gamma}$). Sometimes the map $\boldsymbol{\gamma}'\colon (a,b) \to \mathbf{R}^n$ is also called the velocity vector, even though, rightly speaking, it is a family of velocity vectors. We might indulge in these two types of looseness occassionaly for ease of writing. We say $\boldsymbol{\gamma}$ *passes through* $\boldsymbol{p} \in U$ if there is a $\theta \in (a,b)$ such that $\boldsymbol{\gamma}(\theta) = \boldsymbol{p}$.

(1) Let f and $\boldsymbol{\gamma}$ be as above. Show that

$$\frac{\mathrm{d} f(\boldsymbol{\gamma}(t))}{\mathrm{d}t} = \langle \boldsymbol{v}_{\boldsymbol{\gamma}}(t),\, \boldsymbol{\nabla} f(\boldsymbol{\gamma}(t))\rangle$$

for $t \in (a,b)$.

(2) Let f and $\boldsymbol{\gamma}$ be as above. Suppose S is the hypersurface in U given by the equation $f(\boldsymbol{x}) = c$ where c is a constant. Suppose the path $\boldsymbol{\gamma}$ is actually a path in S, i.e. $\boldsymbol{\gamma}(t) \in S$ for all $t \in (a,b)$. Show that $\boldsymbol{v}_{\boldsymbol{\gamma}}(t)$ is orthogonal to $\boldsymbol{\nabla} f(\boldsymbol{\gamma}(t))$ for all $t \in (a,b)$.

Tangent spaces. Let U be open in $\mathbf{R}^n$ and $\boldsymbol{g}\colon U \to \mathbf{R}^m$ a $\mathscr{C}^1$ map, $\boldsymbol{c} \in \mathbf{R}^m$ a point in the image of $\boldsymbol{g}$ and M the subset of U given by the equation $\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{c}$. Suppose that $\boldsymbol{p} \in M$ is a point such that the rank of $\boldsymbol{g}'(\boldsymbol{p})$ is m. Let $V_{\boldsymbol{p}}$ be the null space of $\boldsymbol{g}'(\boldsymbol{p})$. The space $T_{\boldsymbol{p}} = V_{\boldsymbol{p}} + \boldsymbol{p}$ is called the tangent space to M at $\boldsymbol{p}$.

(3) Show that $V_{\boldsymbol{p}}$ is the space spanned by all velocity vectors $\boldsymbol{v}_{\boldsymbol{\gamma}}(\boldsymbol{p})$ for $\mathscr{C}^1$ paths $\boldsymbol{\gamma}$ taking values in M and passing through $\boldsymbol{p}$. [**Hint:** It is easy to see that the velocity vectors at $\boldsymbol{p}$ of paths in M passing through $\boldsymbol{p}$ lie in $V_{\boldsymbol{p}}$. To show that the space spanned by them is all of $V_{\boldsymbol{p}}$ use the Implicit Function Theorem.]

(4) Let W be an open set in $\mathbf{R}^n$ and $\boldsymbol{f}\colon W \to \mathbf{R}^m$ a $\mathscr{C}^1$ function. Let Γ be the graph of $\boldsymbol{f}$, i.e. $\Gamma = \{(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x})) \in W \times \mathbf{R}^m \mid \boldsymbol{x} \in W\} \subset \mathbf{R}^{m+n}$. Let $\boldsymbol{p} \in W$, and set $A = \boldsymbol{f}'(\boldsymbol{p})$ and $\boldsymbol{b} = \boldsymbol{f}(\boldsymbol{p})$. Show that the tangent space to Γ at $\boldsymbol{q} = (\boldsymbol{p}, \boldsymbol{b}) \in \Gamma$ makes sense (in other words, see that the conditions given for the definition to make sense are satisfied, especially the condition on the rank of a derivative), and show that $T_{\boldsymbol{q}}$ is given by the system of linear equations in $\mathbf{R}^{m+n} = \mathbf{R}^n \times \mathbf{R}^m$:

$$\boldsymbol{y} = A(\boldsymbol{x} - \boldsymbol{p}) + \boldsymbol{b}.$$