HW 3

Due on Jan 27, 2020 (in class).

As before $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$. There are seven questions in this homework assignment, and these are spread over two pages.

Metric spaces. Let (X, d), (Y, δ), (Z, ϱ) be metric spaces. The notion of an open set, closed set, limit points, the closure of a set can be extended to metric spaces in an obvious way, and it is easy to see that if S is a subset of S, then its closure (defined as the union of S and its limit points) is the smallest closed set containing S (the proofs given in Lecture 2 apply *mutatis mutandis* to metric spaces). You may freely use those results.

(1) Let x be a point of X, and $X \xrightarrow{f} Y \xrightarrow{g} Z$ a pair of maps such that f is continuous at x and g is continuous at $f(x)$. Show that $g \circ f$ is continuous at x.

(2) Show that $\{x_n\}$ converges to x in X if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

(3) For a fixed $x_0 \in X$, show that $f\colon X \to \mathbf{R}$ given by $f(x) = d(x, x_0)$ is a continuous function.

Inner Product Spaces. Let V be a vector space over $\mathbf{K}$ and suppose we have a map

$$V \times V \xrightarrow{\langle\, ,\, \rangle} \mathbf{K}$$

such that for x, y, z in V

(i) $\langle x, x\rangle \geq 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

(ii) $\langle x, y\rangle = \overline{\langle y, x\rangle}$ where the bar is complex conjugation.

(iii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ and $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$.

(iv) For $\lambda \in \mathbf{K}$, $\langle \lambda x, y\rangle = \lambda\langle x, y\rangle$ and $\langle x, \lambda y\rangle = \bar{\lambda}\langle x, y\rangle$.

Such a space is called an *inner product space*. Define $\|\ \|\colon V \to [0, \infty)$ by

$$\|x\| = \langle x, x\rangle^{\frac{1}{2}}.$$

(4) Prove the *Cauchy-Schwarz* inequality

$$|\langle a, b\rangle| \leq \|a\|\|b\|$$

for $a, b \in V$. [Hint: Look at §§ 1.1.3 of Lecture 1 for ideas.]

(5) Prove that $\|\ \|$ is a norm on V.

Important inequalities. Recall that a continuous strictly monotone function φ on a closed interval $I = [a, b]$ is one-to-one, $J = \varphi(I)$ is a closed interval say $J = [c, d]$, and the inverse of φ, $\varphi^{-1}\colon J \to I$ is also continuous. Therefore in this case the Riemann integrals $\int_a^b \varphi(x)dx$ and $\int_c^d \varphi^{-1}(x)dx$ make sense and are finite.

(6) Let $f\colon [0, r) \to [0, \infty)$ be continuous and strictly increasing with $f(0) = 0$ (the case $r = \infty$ is allowed). Show that for every a in $[0, r)$ and every b in the image of $[0, r)$ under f, we have

$$ab \leq \int_0^a f(x)dx + \int_0^b f^{-1}(y)dy.$$

(7) Let $p \in (1, \infty)$. Let $q \in (1, \infty)$ be defined by the equation $p^{-1} + q^{-1} = 1$. Show that if a and b are non-negative real numbers, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

(8) (Recreational problem, not to be submitted, but something to think about, and mainly for your entertainment). Show (using the previous problem) that if $p_1, \dots, p_n$ are real numbers in $(1, \infty)$ such that $\sum_{i=1}^n \frac{1}{p_i} = 1$, then for any non-negative real numbers $a_1, \dots, a_n$ we have $a_1 a_2 \dots a_n \leq \sum_{i=1}^n \frac{a_i^{p_i}}{p_i}$. Taking all $p_i = n$, show the usual inequality involving geometric means and arithmetic means.