HW 2

Due on Jan 22, 2020 (in class).

As before $\mathbf{K} \in \{\mathbf{R}, \mathbf{C}\}$.

Normed spaces. Let $(X, \| \, \|)$ be a normed linear space.

(1) Show that $\lim_{n\to\infty} \boldsymbol{x}_n = \boldsymbol{x}$ in X if and only if $\lim_{n\to\infty} \|\boldsymbol{x}_n - \boldsymbol{x}\| = 0$.

(2) Let $S \subset X$ and $f\colon S \to \mathbf{K}$ a map. Show that f is continuous at $\boldsymbol{s} \in S$ if and only if $\lim_{n\to\infty} f(\boldsymbol{s}_n) = f(\boldsymbol{s})$ whenever $\{\boldsymbol{s}_n\}$ is a sequence in S such that $\lim_{n\to\infty} \boldsymbol{s}_n = \boldsymbol{s}$.

(3) Let $S \subset X$ and $\boldsymbol{f}\colon S \to \mathbf{K}^n$ a map, say $\boldsymbol{f} = (f_1, \ldots, f_n)$ where $f_i\colon S \to \mathbf{K}$, $i = 1, \ldots, n$ are the components of $\boldsymbol{f}$. Give $\mathbf{K}^n$ the ℓ^2-norm, i.e., the norm $\| \, \|_2$. Show that $\boldsymbol{f}$ is continuous if and only if each f_i is continuous. [Hint: Use the equivalence of $\| \, \|_2$ and $\| \, \|_\infty$ as well as Problem (2).]

(4) In this problem $\mathbf{K}^n$ and $\mathbf{K}^m$ are each given the Euclidean norm $\| \, \|_2$.

(a) Let $a_1, \ldots, a_n \in \mathbf{K}$. Let $f\colon \mathbf{K}^n \to \mathbf{K}$ be the map given by the formula

$$f(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n \qquad ((x_1, \ldots, x_n) \in \mathbf{K}^n).$$

Prove that f is continuous. [Hint: Use one of the problems above.]

(b) Let $T\colon \mathbf{K}^n \to \mathbf{K}^m$ be a linear transformation of $\mathbf{K}$-vector spaces. Show that T is continuous.

Metric Spaces. Many results about normed spaces are better understood in a more general context.

Definitions.

A) Let X be a non-empty set. A *metric (or a distance function)* on X is a map

$$d\colon X \times X \longrightarrow [0, \infty)$$

such that

(i) $d(x, y) = 0$ if and only if $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) For x, y, z in X,

$$d(x, z) \leq d(x, y) + d(y, z).$$

B) A *metric space* is a pair (X, d) where X is a non-empty set and d is a metric on it.

C) Let (X, d) be a metric space. A sequence $\{x_n\}$ is X is said to converge to $x \in X$ if for every $\epsilon > 0$ there exists $N \in \mathbf{N}$ such that

$$d(x_n, x) < \epsilon \qquad (n \geq N).$$

In this case we write

$$\lim_{n\to\infty} x_n = x.$$

D) A point $a \in X$ is a *limit point* of X if there exists a sequence $\{x_n\}$ in $X \smallsetminus \{a\}$ such that $\lim_{n\to\infty} x_n = a$.

Fix two metric spaces (X, d) and (Y, d') and a map $f\colon X \to Y$.

E) We say the *limit of* $f(x)$ *as* x *approaches* $a \in X$ *is* $y \in Y$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d'(f(x), y) < \epsilon \qquad\qquad (0 < d(x, a) < \delta).$$

(The above is read as : $d'(f(x), y) < \epsilon$ *whenever* $x \in X$ *is such that* $0 < d(x, a) < \delta$.) If y is the limit of $f(x)$ as x approaches a, we write

$$\lim_{x\to a} f(x) = y.$$

F) We say f is *continuous at* $a \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$d'(f(x), f(a)) < \epsilon$$

whenever $x \in X$ satisfies $d(x, a) < \delta$. Let S be a non-empty subset of X. We say f is *continuous on* S if f is continuous at every point of S. We say f is *continuous* if it is continuous on X. We point out that if $a \in X$ is a limit point of X (see (D) above), then f is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$. This is clear from the definitions.

Here are two elementary problems on metric spaces

(5) Let $(X, \|\ \|)$ be a normed linear space and define $d\colon X \times X \to [0, \infty)$ by the formula $d(\boldsymbol{x}, \boldsymbol{y}) = \|\boldsymbol{x} - \boldsymbol{y}\|$. Show that (X, d) is a metric space.

(6) Let (X, d) and (Y, d') be metric spaces and $f\colon X \to Y$ a map.

   (a) Let $a \in X$. Show that $\lim_{x\to a} f(x) = y$ if and only if for every sequence $\{x_n\}$ in $X \smallsetminus \{a\}$ converging to a in X, $\{f(x_n)\}$ converges to y.

   (b) Show that the map f is continuous at $a \in X$ if and only if for every sequence $\{x_n\}$ in X converging to a we have $\lim_{n\to\infty} f(x_n) = f(a)$.

Remark. It may be a good idea to check that the notions of limits, continuity etc., on a normed vector space discussed in §§ 1.2 of Lecture 1 coincide with the same notions for a metric space (see Definitions 1.2.1, 1.2.2, 1.2.3, and 1.2.4 there). You may be asked to verify one or more of these equivalences in a quiz or a test.