HW 1

Due on Jan 15, 2020 (in class).

There are six questions in this Homework, and it spills into two pages. Questions 5 and 6 are on the second page.

Uniform convergence. This section reviews some ideas about uniform convergence.

Definition: Let A be a set and $\{f_n\}$ a sequence of $\mathbf{R}$-valued functions on A. We say $\{f_n\}$ converges *pointwise* on A to a function $f\colon A \to \mathbf{R}$ if for each each $\epsilon > 0$ and each $a \in A$, there exists $N_{a,\epsilon} \in \mathbf{N}$ such that for every $a \in A$ we have

$$|f_n(a) - f(a)| < \epsilon \qquad (n \geq N_{a,\epsilon}).$$

We say $\{f_n\}$ converges *uniformly* to f on A if it converges pointwise to f and the $N_{a,\epsilon}$ above can be chosen to be independent of a. In other words, $\{f_n\}$ converges to f uniformly if for each $\epsilon > 0$, there exists $N = N_\epsilon$, depending only on ϵ, such that for every $a \in A$ we have

$$|f_n(a) - f(a)| < \epsilon \qquad (n \geq N_\epsilon).$$

(1) For $n \in \mathbf{N}$ let $f_n\colon (0,1) \to \mathbf{R}$ be the function given by the formula

$$f_n(x) = \frac{n}{1+nx} \qquad (x \in (0,1)).$$

Show that $\{f_n\}$ is pointwise convergent but not uniformly convergent on $(0,1)$.

(2) Show that $\{f_n\}$, where $f_n\colon [0,1] \to \mathbf{R}$ is the map $f_n(x) = x^n$, is not uniformly convergent.

(3) Come up with a definition of uniform convergence for a sequence of functions $\{f_n\}$ on a set A taking values in a normed linear space W over $\mathbf{R}$. Show that if $A = [0,1]$, and if $\{f_n\}$ is a sequence of continuous W-valued functions on $[0,1]$ which converges uniformly to $f\colon [0,\,1] \to W$, then f is continuous. (You will have to look at the notes on Lecture 1 that I posted to learn the definition of continuous functions from a subset C of a normed space V to a normed space W.)

Normed spaces. Let V and W be normed linear spaces over $\mathbf{R}$ (all the results below work over $\mathbf{C}$ also but I want to keep it simple for this homework).

(4) On $\mathbf{R}^n$ show that $\|\ \|_\infty \leq \|\ \|_2$.

(5) Let $\{\boldsymbol{v}_n\}$ be a sequence in $\mathbf{R}^2$, say $\boldsymbol{v}_n = (x_n, y_n)$. Give $\mathbf{R}^2$ the $\| \, \|_\infty$ norm. Show that $\lim_{n\to\infty} \boldsymbol{v}_n \to \boldsymbol{v}$ if and only if $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$ where $\boldsymbol{v} = (x, y)$.

(6) Use what you know from analysis on $\mathbf{R}$ to come up with a definition of a Cauchy sequence in V. When would you say V is complete? Is $\mathbf{R}^n$ with $\| \, \|_\infty$ complete?