Locally free sheaves and vector bundles

This is meant to help with future lectures.

Basics: Given a ring A and $M \in Mod_A$, we have the (non-commutative) graded **tensor A-algebra** $T^\bullet(M) = \bigoplus_{n \geq 0} T^n(M)$ with $T^n(M) = \underbrace{M \otimes_A \cdots \otimes_A M}_{n\text{-times}}$. The product $T^n(M) \otimes_A T^m(M) \longrightarrow T^{n+m}(M)$ is obvious. Modding out by the two sided ideal I in $T^\bullet(M)$ generated by $\{ m \otimes m' - m' \otimes m \mid m, m' \in M \}$, we obtain a commutative A-algebra

$$Symm^\bullet(M) = T^\bullet(M)/I$$

If B is a (commutative) A-algebra and $M \xrightarrow{\phi} B$ an A-module map, then we have a unique map of A-algebras $\Psi : Symm^\bullet(M) \longrightarrow B$ such that $\Psi(m) = \phi(m)$, $m \in M$.

Important example: Let F be a free A-module of rank n. The co-ordinate free way of defining "polynomial functions" on F is as elements of $P(F) := Symm^\bullet(F^*)$, where F^* is the dual of F, i.e. $F^* = Hom_A(F, A)$. Indeed given $p \in P(F)$, then $p = \sum_{\text{finite sum}} a_{i_1 \ldots i_d} \alpha_{i_1} \cdots \alpha_{i_d}$, $a_{i_1 \ldots i_d} \in A$, $\alpha_{i_j} \in F^*$, the product $\alpha_{i_1} \cdots \alpha_{i_n}$ the image of $\alpha_{i_1} \otimes \cdots \otimes \alpha_{i_n}$ in $P(F)$; and hence we have, for $v \in F$

$$p(v) := \sum a_{i_1 \ldots i_d} \alpha_{i_1}(v) \cdots \alpha_{i_d}(v) \in A.$$

If we choose a basis $f_1, \ldots, f_n$ for F, and $f_1^*, \ldots, f_n^*$ the dual basis for F^* (i.e. $f_i^*(f_j) = \delta_{ij}$, $1 \leq i, j \leq n$) then $P(F)$ gets identified with $A[T_1, \ldots, T_n]$ with

$$\sum_i a_{\nu_1\ldots\nu_n} (f_1^*)^{\nu_1} \cdots (f_n^*)^{\nu_n} \longmapsto \sum a_{\nu_1\ldots\nu_n} T_1^{\nu_1}\cdots T_n^{\nu_n}$$

Under this identification, if $v = a_1 f_1 + \cdots + a_n f_n$, then $p(v) = q(a_1,\ldots,a_n)$, for $p \in P(F)$ and q its image in $A[Y_1,\ldots,Y_n]$.

This coordinate free way of looking at polynomials on F is useful.

Projective modules: Recall that finitely presented projective modules are locally free (more is true, but we will not go into that – look up theorem of Kaplansky on projective modules on local rings). Thus if P is a projective A-module, and is finitely presented (i.e. we have an exact sequence $A^e \longrightarrow A^d \longrightarrow P \longrightarrow 0$, d, e non-neg integers) then $\exists\, a_1,\ldots,a_m \in A$ s.t. P_{a_i} is a free A_{a_i} module and $\langle a_1,\ldots,a_m\rangle = A$. Since $\mathrm{Sym}^\bullet$ is functorial and behaves well with respect to localisation, if $U_i := D(a_i) \subset \operatorname{Spec} A$, then $(\mathrm{Sym}^\bullet(P^*))_{a_i} \cong A_{a_i}[T_1,\ldots,T_n]$.

Vector bundles: Let P be as above. Let $V = \operatorname{Spec}(\mathrm{Sym}^\bullet P^*)$. Then we have a map

$$\pi : V \longrightarrow \operatorname{Spec} A$$

corresponding to the A-algebra $A \longrightarrow \mathrm{Sym}^\bullet(P)$.

From what we said above

$$\pi^{-1}(U_i) \cong \mathbb{A}^n_{U_i}, \quad i = 1,\ldots,m.$$

Moreover, the transition functions on $U_{ij} = U_i \cap U_j$ are linear automorphisms, since all they involve is a change of basis of the free $A_{a_i a_j}$-module $P_{a_i a_j}$.

In other words $V \xrightarrow{\pi} \operatorname{Spec} A$ is a vector bundle.

This will be defined more formally in the course, but you can get ahead of the game by looking up the defn.

This generalises to locally free sheaves on a scheme X. Let $\mathcal{P}$ be a locally free $\mathcal{O}_X$ module, i.e., $\exists$ an open cover $\{U\}$ of X s.t. $\mathcal{P}|_U \cong \mathcal{O}_U^n$. Set $\mathcal{B} = \operatorname{Sym}^{\bullet}_{\mathcal{O}_X}(\mathcal{P}^*)$, where $\operatorname{Sym}^{\bullet}(\mathcal{P}^*)$ has an obvious meaning. Then we have an X-scheme

$$\mathcal{V} := \underline{\operatorname{Spec}}(\mathcal{B})$$

as in HW-6. Let

$$\pi : \mathcal{V} \longrightarrow X$$

be the structure map. Then $\mathcal{V} \longrightarrow X$ is a vector bundle on X.

Invertible sheaves: Locally free sheaves of rank 1 are called invertible sheaves. We mentioned this in class. The corresponding vector bundles are called line bundles.