DIRECT IMAGES OF QUASI-COHERENT SHEAVES

The symbol $\mathcal{Ab}$ will denote the category of abelian groups and $\mathbb{S}\mathrm{ch}$ the category of schemes. If X is a topological space, $\mathcal{Psh}_X$ and $\mathcal{Sh}_X$ denote the category of presheaves and the category of sheaves respectively on X. By a ring we mean a commutative ring with identity. For a ring A, Mod_A denotes the category of A-modules. For a sheaf of rings $\mathscr{A}$ on a topological space, $\mathrm{Mod}_{\mathscr{A}}$ will denote the category of $\mathscr{A}$-modules.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Separated and quasi-compact maps

For any continuous map $f\colon X \to Y$ of topological spaces, the right derived functors $\mathrm{R}^i f_*\colon \mathcal{Sh}_X \to \mathcal{Sh}_Y$ are called the *higher direct images* of f. This includes the case $i = 0$, i.e. we regard $f_* = \mathrm{R}^0 f_*$ also as a higher direct image. Let me remind you that if $\mathscr{F} \in \mathcal{Sh}_X$, then $\mathrm{R}^i f_*\mathscr{F} = \mathrm{H}^i(f_*\mathscr{I}^\bullet)$ where $\mathscr{I}^\bullet$ is a bounded below injective resolution of $\mathscr{F}$.

1.1. **Higher direct images of flasque sheaves.** Let $f\colon X \to Y$ be as above. If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence of flasque sheaves on X, it is clear (by testing on open subsets of Y) that $0 \to f_*\mathscr{F} \to f_*\mathscr{G} \to f_*\mathscr{H} \to 0$ is exact. Since injectives are flasque, it then follows (from the formula for $\mathrm{R}^i f_*$ given above and using classical injective resolutions), that $\mathrm{R}^i f_*\mathscr{F} = 0$ for $i \geq 1$ if $\mathscr{F}$ is flasque. Thus flasque sheaves are f_*-acyclic, and hence can be used to compute higher direct images.

If X and Y are ringed spaces, and $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_X}$, then it is easy to see that $\mathrm{R}^i f_*\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_Y}$. Indeed,we can take a bounded below resolution $\mathscr{F} \to \mathscr{I}^\bullet$ by injectives in $\mathrm{Mod}_{\mathscr{O}_X}$, and this will be a flasque resolution. The rest is straightforward, since this observation implies that $\mathrm{R}^i f_*\mathscr{F} \xrightarrow{\sim} \mathrm{H}^i(f_*\mathscr{I}^\bullet)$.

1.2. **Higher direct images of quasi-coherent sheaves.** Recall that a map of schemes $f\colon X \to Y$ is *quasi-compact* if $f^{-1}(V)$ is quasi-compact for every quasi-compact open subset V of Y.

Theorem 1.2.1. *Let $f\colon X \to Y$ be quasi-compact and separated map and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module.*

(a) $\mathrm{R}^i f_*\mathscr{F}$ *is quasi-coherent for every* $i \geq 0$.
(b) *If Y is affine, say $Y = \mathrm{Spec}\, A$, and if for each $i \geq 0$, M_i is the A-module $\mathrm{H}^i(X, \mathscr{F})$, then we have a canonical isomorphism $\mathrm{R}^i f_*\mathscr{F} \xrightarrow{\sim} \widetilde{M_i}$ for every $i \geq 0$*

Proof. The basic idea is straightforward. Part (a) is local on the base, and so we assume, without loss of generality, that $Y = \mathrm{Spec}\, A$. In slightly greater detail, the restriction of a flasque sheaf to an open subset is also flasque, whence higher direct images behave well with respect to restrictions to open subsets of the base, i.e.

$\mathrm{R}^i(f_V)_*(\mathscr{F}|_{f^{-1}(V)}) = \mathrm{R}^i f_*\mathscr{F}|_V$ for every open set V in Y, where $f_V\colon f^{-1}(V) \to V$ is the obvious map induced by f.[1]

We begin with two observations. Since Y is affine, it is quasi-compact and separated. Therefore, since f is quasi-compact and separated, X is quasi-compact and separated. Since X is quasi-compact it has a finite cover $\mathfrak{U} = \{U_0, \ldots, U_d\}$ by affine open subschemes. With $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ as usual denoting the sheaf Čech complex, we will show

- **(i)** The standard resolution $\mathscr{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ of $\mathscr{F}$ is an f_*-acyclic resolution of $\mathscr{F}$, so that $\mathrm{R}^i f_*\mathscr{F} \cong \mathrm{H}^i(\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}))$.
- **(ii)** The $\mathscr{O}_Y$-modules $f_*\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$, $0 \le p \le d$, are all quasi-coherent. This would prove that $\mathrm{H}^i(\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}))$, and hence by **(i)**, $\mathrm{R}^i f_*\mathscr{F}$ is quasi-coherent.

To that end we introduce some notations. For $0 \le p \le d$, we write $\boldsymbol{i}$ for $(i_0, \ldots, i_p) \in \mathbf{Z}^p$, with $0 \le i_0 < \ldots i_p \le d$, and for such $\boldsymbol{i}$, we set $U_{\boldsymbol{i}} = U_{i_0} \cap \cdots \cap U_{i+p}$. Since X is separated , by Proposition 2.1.4 of Lecture 13, $U_{\boldsymbol{i}}$ is an affine open set. Let $j^{\boldsymbol{i}}\colon U_{\boldsymbol{i}} \to X$ be the inclusion map. Then

$$(*) \qquad \mathscr{C}^p(\mathfrak{U}, \mathscr{F}) = \bigoplus_{0 \le i_0 < \cdots < i_p \le d} j^{\boldsymbol{i}}_*(\mathscr{F}|_{U_{\boldsymbol{i}}}).$$

Now $f_*j^{\boldsymbol{i}}_*(\mathscr{F}|_{U_{\boldsymbol{i}}}) = (f \circ j^{\boldsymbol{i}})_*(\mathscr{F}|_{U_{\boldsymbol{i}}})$, and $f \circ j^{\boldsymbol{i}}\colon U_{\boldsymbol{i}} \to Y$ is an affine map, being a map between affine schemes. It follows that $(f \circ j^{\boldsymbol{i}})_*(\mathscr{F}|_{U_{\boldsymbol{i}}})$ is quasi-coherent by item **4** of § 3.1.1 of Lecture 13. Thus $f_*\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is quasi-coherent. This proves **(ii)**.

We now prove **(i)**. If $j\colon U \to X$ is an open immersion, with U affine, then we have seen earlier that in this case j is affine, since $j^{-1}(V) = U \cap V$ is affine for every affine open subscheme V of the separated scheme X (see Proposition 2.1.4 of Lecture 13). Let $\mathscr{G}$ be quasi-coherent on U. Since j is affine, $j_*\mathscr{G}$ is quasi-coherent on X (see item **4** of § 3.1.1 of Lecture 13). Moreover, from Theorem 3.1.2 of Lecture 13 we see that $j_*\colon \mathrm{Qcoh}(U) \to \mathrm{Qcoh}(X)$ is exact and $\mathrm{R}^n j_*\mathscr{G} = 0$ for $n \ge 1$ (since, by our choice, $\mathscr{G} \in \mathrm{Qcoh}(U)$). Now suppose $\mathscr{G} \to \mathscr{I}^\bullet$ is a classical resolution of $\mathscr{G}$ by injective objects in $\mathrm{Mod}_{\mathscr{O}_U}$. From the above observations, $j_*\mathscr{I}^\bullet$ is a flasque resolution of $j_*\mathscr{G}$. It follows that $\mathrm{R}^n f_*(j_*\mathscr{G}) \cong \mathrm{H}^n(f_*j_*\mathscr{I}^\bullet) = \mathrm{H}^n((f \circ j)_*\mathscr{I}^\bullet) = \mathrm{R}^n(f \circ j) * \mathscr{G}$. Now $f \circ g\colon U \to Y$ is a map of affine schemes, and hence is an affine. One more application of Theorem 3.1.2 of Lecture 13 gives that $\mathrm{R}^n(f \circ g)_*\mathscr{G} = 0$ for $n \ge 1$, i.e., from the string of isomorphisms we just wrote, $\mathrm{R}^n f_*(j_*\mathscr{G}) = 0$ for $n \ge 1$. In other words $j_*\mathscr{G}$ is f_*-acyclic. In particular, the summand $j^{\boldsymbol{i}}_*(\mathscr{F}|_{U_{\boldsymbol{i}}})$ in the direct sum decomposition of $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ in $(*)$ is f_*-acyclic. This proves **(i)**. We have therefore proved (a) of the Theorem.

For (b) we use the cover $\mathfrak{U}$ above and continue to use the notations we have introduced above. Since Y is affine, $\Gamma(Y, -)\colon \mathrm{Qcoh}(Y) \to \mathrm{Mod}_A$ and $\widetilde{(-)}\colon \mathrm{Mod}_A \to \mathrm{Qcoh}(Y)$ are pseudo-inverses to each other, and both are exact functors. It is therefore enough to show that $\Gamma(Y, \mathrm{R}^i f_*\mathscr{F}) \xrightarrow{\sim} M_i$. This follows from the following

[1]The statement about the higher direct image being well-behaved with respect to restrictions to open subsets on the base is a topoliogical statement and not a scheme-theoretic one; so just for that statement X and Y are topoligical spaces and f a continuous map.

sequence of isomorphisms (explanations after the displayed sequence of isomorphisms):

$$\begin{aligned}\Gamma(Y,\,\mathrm{R}^i f_*\mathscr{F}) &\xrightarrow{\ \sim\ } \Gamma(Y,\,\mathrm{H}^i(\mathscr{C}^\bullet(\mathfrak{U},\,\mathscr{F})))\\ &\xrightarrow{\ \sim\ } \mathrm{H}^i(\Gamma(Y,\,\mathscr{C}^\bullet(\mathfrak{U},\,\mathscr{F})))\\ &=\!=\, \mathrm{H}^i(C^\bullet(\mathfrak{U},\,\mathscr{F}))\\ &\xrightarrow{\ \sim\ } \mathrm{H}^i(X,\,\mathscr{F}).\end{aligned}$$

The fist isomorphism is from the statement in **(i)**. The second isomorphism follows from the fact that $\Gamma(Y,\,-)$ is exact, whence commutes with cohomology. The last isomorphism follows from Theorem 3.2.4 of of Lecture 13, since X is separated. ☐

2. Quasi-compact and quasi-separated maps

We give a more advanced version of Theorem 1.2.1 in this section. Our proofs will largely be sketches, but can easily be filled out by anyone who has followed the course so far.

2.1. **$\mathscr{B}$-sheaves again.** Let X be a topological space, $\mathscr{B}$ a basis for the topology closed under pairwise intersections, and F a presheaf on X such that $F|_{\mathscr{B}}$ is a $\mathscr{B}$-sheaf. Let $\mathscr{F}$ be the unique extension of $F|_{\mathscr{B}}$ to a sheaf on X. Recall, that $\mathscr{F}$ is the unique sheaf on X such that $\mathscr{F}(U) = F(U)$ for all $U \in \mathscr{B}$. Then it is easy to see that $\mathscr{F} = F^+$. To begin with, by Problem 10 of Homework 1 we get a map $F \to \mathscr{F}$. Now suppose $F \to \mathscr{G}$ is a map of presheaves with $\mathscr{G}$ a sheaf. Then for any open set U in X we have a cover (U_α) of U by members of $\mathscr{B}$, and using the formula $(*)$ in Problems 6–9 of Homework 1 or problem 10 of *loc.cit.*, we see that we have a map $\mathscr{F}(U) \to \mathscr{G}(U)$ which, as U varies over open sets in X, gives us a map of sheaves, and that this map is the unique one such that $F \to \mathscr{G}$ factors as $F \to \mathscr{F} \to \mathscr{G}$. The details are left to you. We record the result below

Proposition 2.1.1. *Let X be a topological space, $\mathscr{B}$ a basis for the topology on X which is closed under pairwise intersection, and F a presheaf on X such that $F|_{\mathscr{B}}$ is a $\mathscr{B}$-sheaf. Let $\mathscr{F}$ be the unique sheaf on X such that $\mathscr{F}|_{\mathscr{B}} = F|_{\mathscr{B}}$. Then $\mathscr{F} = F^+$, the sheafification of the presheaf F.*

2.2. **Direct limit of sheaves.** Let $(\mathscr{F}_\gamma)_{\gamma\in\Gamma}$ be a direct system of sheaves on a topological space X. Let $P = P_{(\mathscr{F}_\gamma)}$ be the presheaf

$$U \rightsquigarrow \varinjlim_{\gamma} \Gamma(U,\,\mathscr{F}_\gamma).$$

Then the sheaficiation P^+ has the universal property of direct limits in the category of sheaves, a fact that is easily checked. This proves that direct limits exist in the category $\mathcal{Sh}_X$. Moreover the natural map $P \to P^+$ translates to maps, one for each open set U in X

$$\varinjlim_{\gamma} \Gamma(U,\,\mathscr{F}_\gamma) \longrightarrow \Gamma(U,\,\varinjlim_{\gamma} \mathscr{F}_\gamma). \tag{2.2.1}$$

2.3. **Characterising quasi-compact maps.** A map $f\colon X \to Y$ is said to be *quasi-separated* if the diagonal map $\Delta_{X/Y}\colon X \to X \times_Y X$ is a quasi-compact map.

We wish to give some characterisations of quasi-compact maps. The following list should suffice for this course.

Proposition 2.3.1. *Let $f\colon X \to Y$ be a map of schemes. The following are equivalent*

(a) *f is quasi-compact.*
(b) *For every affine open subcheme V of Y, $f^{-1}(V)$ is quasi-compact.*
(c) *There is a cover $\mathfrak{V}$ of Y by affine open subschemes such that $f^{-1}(V)$ is quasi-compact for every $V \in \mathfrak{V}$.*

Proof. It is clear that (a) $\Rightarrow$ (b) $\Rightarrow$ (c). We now prove (b) $\Rightarrow$ (a). Suppose (b) is true and W is a quasi-compact open in Y. Then we can cover W by a finite collection of affine opens $V_1, \dots, V_n$. For each $i \in \{1, \dots, n\}$, $f^{-1}(V_i)$ is quasi-compact, whence covered by a finite collection of affine opens U_{ij}. The finite collection $\{U_{ij}\}$ covers $f^{-1}(W)$ and hence $f^{-1}(W)$ is the finite union of quasi-compact open sets. It follows that $f^{-1}(W)$ is quasi-compact, giving us (a).

We are thus done if we prove (c) $\Rightarrow$ (b). Assume that we have a cover $\mathfrak{V}$ of Y by affine open subschemes such that $f^{-1}(V)$ is quasi-compact for every $V \in \mathfrak{V}$. Let W be an affine open subset of Y. For each $V \in \mathfrak{V}$, we can find an open cover of $\mathfrak{W}_V$ of $W \cap V$ by standard affine open subschemes of V, i.e. a cover of $W \cap V$ by sets of the form V_g. Then $\cup_{V \in \mathfrak{V}} \mathfrak{W}_V$ is an affine open cover of W. It has a finite subcover $\mathfrak{W} = \{W_1, \dots, W_n\}$. Let $V_1, \dots, V_n \in fV$ be such that $W_i \subset W \cap V_i$ and W_i is a standard open subset of V_i. Fix $i \in \{1, \dots, n\}$. Since $f^{-1}(V_i)$ is quasi compact, it can be covered by a finite number of affine open sets $\{U_{ij}\}$. If $W_i = (V_i)_g$, then clearly $f^{-1}(W_i) \cap U_{ij} = (U_{ij})_g$, whence $f^{-1}(W_i) \cap U_{ij}$ is a standard open subset of U_{ij}. It follows that $f^{-1}(W_i) \cap U_{ij}$ is affine. Now $f^{-1}(W)$ is clearly the union of $f^{-1}(W_i) \cap U_{ij}$ as i and j vary, and hence it is a finite union. It follows that $f^{-1}(W)$ is quasi-compact, being the finite union of affine open subschemes. $\square$

Corollary 2.3.2. *Let*

$$\begin{array}{ccc} W & \xrightarrow{v} & X \\ {\scriptstyle g}\downarrow & \square & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[u]{} & Y \end{array}$$

be a cartesian square. If f is quasi-compact, so is g. In particular, if Y is affine and X and Z are quasi-compact, then so is $W = X \times_Y Z$.

Proof. We will repeatedly use the following: If $f\colon X \to Y$ is a map of schemes with Y affine, then f is quasi-compact if and only if X is quasi-compact. This follows from (c) of Proposition 2.3.1. Now, let f be quasi-compact. We can cover Z be affine opens whose image under u lie in an affine open of Y. So we are reduced to the case where Y and Z are affine. Since f is quasi-compact and Y is affine, X is quasi-compact and can be covered by a finite number of affines $\{U_i\}$. Then $v^{-1}(U_i) = U_i \times_Y Z$ is affine, and so we can cover W by a finite number of affines. Thus W is quasi-compact. By our observation in the first line of this proof, g is quasi-compact. The rest of the corollary also follows from that first line and the fact that quasi-compact maps behave well under base change. $\square$

Corollary 2.3.3. *Let $f\colon X \to Y$ be a quasi-separated map, with Y affine. Then the intersection of any two quasi-compact open subschemes of X is again quasi-compact. Moreover, X is a quasi-separated scheme.*

Proof. Let U and V be quasi-compact open subschemes of X. Since Y is affine, by Corollary 2.3.2, $U \times_Y V$ is quasi-compact. Since $U \cap V = \Delta_{X/Y}^{-1}(U \times_Y V)$, and $\Delta_{X/Y}$ is quasi-compact, and hence $U \cap V$ is quasi-compact.

It remains to show that X is quasi-separated. Let $\mathfrak{U}$ be a cover of X by affine open subschemes. Then $\{U \times_{\mathbf{Z}} V\}_{U,V \in \mathfrak{U}}$ is an affine open cover of $X \times_{\mathbf{Z}} X$. Since $\Delta_{X/\mathbf{Z}}^{-1}(U \times_{\mathbf{Z}} V) = U \cap V$, which we saw was quasi-compact, by (c) of Proposition 2.3.1 we see that $\Delta_{X/\mathbf{Z}} \colon X \to X \times_{\mathbf{Z}} X$ is a quasi-compact map. Thus X is quasi-separated. $\square$

2.3.4. We have seen from Proposition 2.3.1 (c) and from Corollary 2.3.3 that if $f \colon X \to Y$ is a map of schemes with Y affine, then X is quasi-compact (respectively quasi-separated) whenever $X \to Y$ is quasi-compact (respectively quasi-separated). These statements are often abbreviated as ' 'X is quasi-compact if and only if it is quasi-compact over an affine" and "X is quasi-separated if and only if it is quasi-separated over an affine".

On consequence is the following important result.

Proposition 2.3.5. *Let X be quasi-compact and quasi-separated, $(\mathscr{F}_\gamma)_{\gamma \in \Gamma}$ a direct system of sheaves on X, and U a* quasi-compact *open subscheme of X. Then*

$$\varinjlim_{\gamma} \Gamma(U, \mathscr{F}_\gamma) \xrightarrow{\sim} \Gamma(U, \varinjlim_{\gamma} \mathscr{F}_\gamma),$$

where the map underlying the isomorphism is (2.2.1).

Proof. Let $\mathscr{B}$ be the collection of quasi-compact open subschemes of X. Since $\mathscr{B}$ contains all affine open subschemes, it is a basis for the topology on X. Moreover by Corollary 2.3.3, $\mathscr{B}$ is closed under pairwise intersection.

Let P be the presheaf defined in § 2.2, i.e. the presheaf $V \rightsquigarrow \varinjlim_{\gamma} \Gamma(V, \mathscr{F}_\gamma)$. We claim that $P|_{\mathscr{B}}$ is a $\mathscr{B}$-sheaf. Suppose we prove this. Then by Proposition 2.1.1 we are done, for, according to *loc.cit.*, if this were true, P^+ would agree with P on members of $\mathscr{B}$, which is exactly the assertion of our proposition.

Let us prove that $P|_{\mathscr{B}}$ is a $\mathscr{B}$-sheaf. To that end, let V be a quasi-compact set. Suppose we have an open cover $\{U_i\}_{i \in I}$ of V with each $U_i \in \mathscr{B}$ and sections $s_i \in P(U_i)$, $i \in I$ which satisfy the usual gluing conditions for a sheaf. Now P is a presheaf, and we cannot *a priori* expect gluing of the s_i. However, since U is quasi-compact, we have a finite subcover of $\{U_i\}$, and we replace our original cover by this finite subcover, which we relabel as $\{U_1, \dots, U_n\}$. For $i, j \in \{1, \dots, n\}$ we have $s_i|_{U_{ij}} = s_j|_{U_{ij}}$, where $U_{ij} := U_i \cap U_j$. Now there exists $\gamma_i \in \Gamma$ and $t_i \in \mathscr{F}_{\gamma_i}(U_i)$ such that t_i maps to s_i under the natural map $\Gamma(U_i, \mathscr{F}_{\gamma_i}) \to \varinjlim_{\gamma} \Gamma(U_i, \mathscr{F}_\gamma) = P(U_i)$. Since we have only a finite number of γ_i to deal with, we can find $\gamma' \in \Gamma$ such that $\gamma_i \prec \gamma'$ for $i = 1, \dots, n$. Let t'_i be the image of t_i in $\mathscr{F}_{\gamma'}(U_i)$. Then on U_{ij}, $t'_j - t'_i$ maps to zero in $\varinjlim_{\gamma} \Gamma(U_{ij}, \mathscr{F}_\gamma) = P(U_{ij})$. This means that there is a $\gamma'' \in \Gamma$ with $\gamma' \prec \gamma''$, such that if t''_i is the image of t'_i in $\Gamma(U_i, \mathscr{F}_{\gamma''})$, then the collection (t''_i) satisfies gluing conditions for the cover $\{U_1, \dots, U_n\}$ of U. Since $\mathscr{F}_{\gamma''}$ is a sheaf, this means there exits a unique $t \in \Gamma(V, \mathscr{F}_{\gamma''})$ whose restrictions to each U_i is t''_i. It is easy to see that is $s \in P(V)$ is the image of t, then $s|_{U_i} = s_i$ for $i = 1, \dots, n$. We leave it to the reader to check that this proves that $P|_{\mathscr{B}}$ is a $\mathscr{B}$-sheaf. $\square$

2.4. **Quasi-flasque sheaves on quasi-compact quasi-separated schemes.** In [K], Kempf gave the following definition of quasi-flasque sheaves for spaces more general than what we are considering, and he called these sheaves *quasi-flabby* *sheaves*. The terms "flabby" and "flasque" are synonymous in sheaf theory, and since we have used flasque, for consistency, we continue to do so.

Definition 2.4.1. Let X be a quasi-compact quasi-separated scheme, and $\mathscr{F}$ a sheaf on X. $\mathscr{F}$ is said to be *quasi-flasque* if $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective for every quasi-compact open subset of X.

The analogues of parts (a) and (b) of Proposition 3.1.3 of Lectures 10 and 11 hold without any change in the proof. If anything (a) is simpler in the case of quasi-flasque sheaves, since one can avoid Zorn's Lemma using finite covers. For the record, the statements in our case are as follows: Let

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

be an exact sequence of sheaves on X.

(a) If $\mathscr{F}$ is quasi-flasque and U is a *quasi-compact* open subset of X then

$$0 \longrightarrow \mathscr{F}(U) \longrightarrow \mathscr{G}(U) \longrightarrow \mathscr{H}(U) \longrightarrow 0$$

is also exact.

(b) If $\mathscr{F}$ and $\mathscr{G}$ are quasi-flasque, then so is $\mathscr{H}$.

In the proof of the above items, the fact that $U \cap V$ is quasi-compact whenever U and V are quasi-compact and open plays an important role, as you will see when you transfer the proofs in Lectures 10–11 to the above situation. In other words, the quasi-separated hypothesis is important.

The analogue of Lemma 3.1.4 of Lectures 10 and 11 also holds for quasi-flasque sheaves, and the proof is identical since X is quasi-compact.

Now flasque sheaves are clearly quasi-flasque. If $\mathscr{F}$ is quasi-flasque, and $\mathscr{F} \to \mathscr{G}^\bullet$ is a classical resolution by flasque sheaves, then $0 \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{G}^\bullet)$ is exact by the just mentioned analogue to Lemma 3.1.4 in Lectures 10–11. It follows that $\mathrm{H}^i(X, \mathscr{F}) \cong \mathrm{H}^i(\Gamma(X, \mathscr{G}^\bullet)) = 0$ for $i \geq 1$. We have thus proved:

Proposition 2.4.2. *If X is quasi-compact and quasi-separated and $\mathscr{F}$ a quasi-flasque sheaf on X, then $\mathscr{F}$ is acyclic.*

This means that on quasi-compact quasi-separated schemes, resolutions by quasi-flasque sheaves can compute cohomology of sheaves.

Lemma 2.4.3. *Let X be a quasi-compact and quasi-separated scheme, Let $(\mathscr{F}_\gamma)$ be a direct system of quasi-flasque sheaves on X. Then $\varinjlim_\gamma \mathscr{F}_\gamma$ is also quasi-flasque.*

Proof. This follows from Proposition 2.3.5 and the fact that on direct systems of abelian groups, direct limit is an exact functor and hence preserves surjections of abelian groups. ☐

2.4.4. If $(\mathscr{G}_\gamma^\bullet)$ is a direct system of sheaves on X with X quasi-compact and quasi-separated, and each $\mathscr{G}_\gamma^\bullet$ is exact, then $\varinjlim_\gamma \mathscr{G}_\gamma^\bullet$ is also exact. The idea is to show exactness at the level of stalks. For this check that

$$\varinjlim_\gamma \varinjlim_{U \ni x} \Gamma(U, \mathscr{G}_\gamma^p) = \varinjlim_{U \ni x} \varinjlim_\gamma \Gamma(U, \mathscr{G}_\gamma^p)$$

where U runs over quasi-compact open neighbourhoods of x. We leave the details to the reader.

The following is a generalisation of the result in [H, Chapter III, Lemma 2.18, p.209], in the sense that we do not need our underlying topological space to be noetherian.

Theorem 2.4.5. *Let X be a quasi-compact, quasi-separated scheme and $(\mathscr{F}_\gamma)$ a direct system of sheaves on X. Then for each $i \geq 0$ we have a canonical isomorphism*

$$\varinjlim_\gamma \mathrm{H}^i(X, \mathscr{F}_\gamma) \xrightarrow{\sim} \mathrm{H}^i(X, \varinjlim_\gamma \mathscr{F}_\gamma)$$

where the map underlying the isomorphism is the natural map arising from the universal property of direct limits.

Proof. Let

$$0 \longrightarrow \mathscr{F}_\gamma \longrightarrow G^0(\mathscr{F}_\gamma) \longrightarrow \ldots \longrightarrow G^n(\mathscr{F}_\gamma) \longrightarrow \ldots$$

be the Godement resolution of $\mathscr{F}$. Then, using our remark in 2.4.4 and Lemma 2.4.3, we see that $\varinjlim_\gamma G^\bullet(\mathscr{F}_\gamma)$ is a quasi-flasque resolution of $\varinjlim_\gamma \mathscr{F}_\gamma$. We then have (explanations after the displayed sequence of isomorphisms):

$$\begin{aligned} \varinjlim_\gamma \mathrm{H}^i(X, \mathscr{F}_\gamma) &\xrightarrow{\sim} \varinjlim_\gamma \mathrm{H}^i(\Gamma(X, G^\bullet(\mathscr{F}_\gamma))) \\ &\xrightarrow{\sim} \mathrm{H}^i(\varinjlim_\gamma \Gamma(X, G^\bullet(\mathscr{F}_\gamma))) \\ &\xrightarrow{\sim} \mathrm{H}^i(\Gamma(X, \varinjlim_\gamma G^\bullet(\mathscr{F}_\gamma))) \\ &\xrightarrow{\sim} \mathrm{H}^i(X, \varinjlim_\gamma \mathscr{F}_\gamma) \end{aligned}$$

The second isomorphism arises from the fact that on direct systems of abelian groups, cohomology commutes with $\varinjlim$. The third is from Proposition 2.3.5 and the fact that X is quasi-compact. The last is from Lemma 2.4.3 and Proposition 2.4.2. □

2.4.6. We know from Problem 3 of Homework 3 that if X is affine, say $X = \operatorname{Spec} R$, and t an element in R, then for any A-module Q, if (Q_n) is the direct system which has $Q_n = Q$, for evert $n \geq 0$, and whose transition maps $\mu_{m,n} \colon Q_m \to Q_n$ for $m \leq n$ is multiplication by t^{n-m}, then $\varinjlim_n Q_n = Q_t$. in terms of quasi-coherent sheaves, one can interpret this as follows. Let $\mathscr{F} = \widetilde{Q}$ and $(\mathscr{F}_n)$ the direct system of quasi-coherent sheaves corresponding to (Q_n), so that $\mathscr{F}_n = \mathscr{F}$ for every $n \geq 0$ and the transition maps from $\mathscr{F}_m \to \mathscr{F}_n$ are multiplication by t^{n-m} for $n \geq m$. Then

$$\varinjlim_n \mathscr{F}_n = {}_{D(t)}\mathscr{F},$$

where ${}_U\mathscr{F}$ for an open subset U of X has its usual meaning, namely $j_*(\mathscr{F}|_U)$, with $j \colon U \to X$ the inclusion.

Here is a generalisation of the result in the above remark:

Lemma 2.4.7. *Let X be quasi-compact and quasi-separated, $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module, and $g \in \Gamma(X, \mathscr{O}_X)$. Let $(\mathscr{F}_n)$ be the direct system of quasi-coherent sheaves with $\mathscr{F}_n = \mathscr{F}$ for all $n \geq 0$ and for $n \geq m$, the transition map $\mathscr{F}_m \to \mathscr{F}_n$ is multiplication by g^{n-m}. Then*

$$\varinjlim_n \mathscr{F}_n = {}_{X_g}\mathscr{F}.$$

Proof. Let U be a quasi-compact open subscheme of X, $\mathfrak{U} = \{U_1, \ldots, U_d\}$ a finite affine open cover of U and for $1 \leq \alpha, \beta \leq d$, $\mathfrak{V}_{\alpha\beta} = \{V_{\alpha\beta\gamma}\}_\gamma$ a finite affine open cover of $U_{\alpha\beta} := U_\alpha \cap U_\beta$. For each $n \geq 0$, we have an exact sequence

$$(\flat)_n \qquad 0 \longrightarrow \Gamma(U, \mathscr{F}_n) \longrightarrow \bigoplus_{\alpha=1}^{d} \Gamma(U_\alpha, \mathscr{F}_n) \longrightarrow \bigoplus_{\alpha\beta\gamma} \Gamma(V_{\alpha\beta\gamma}, \mathscr{F}_n).$$

Now, for any two open sets V and W of X, $({}_V\mathscr{F})|_W = {}_{V\cap W}(\mathscr{F}|_W)$. From this and the remark in 2.4.6 we see that upon applying $\varinjlim_n$ to the exact sequence $(\flat)_n$ we get an exact sequence

$$(\varinjlim_n (\flat)_n) \qquad 0 \longrightarrow \varinjlim_n \Gamma(U,\, \mathscr{F}_n) \longrightarrow \bigoplus_{\alpha=1}^{d} \Gamma(U_\alpha,\, {}_{X_g}\mathscr{F}) \longrightarrow \bigoplus_{\alpha\beta\gamma} \Gamma(V_{\alpha\beta\gamma},\, {}_{X_g}\mathscr{F}).$$

Thus $\varinjlim_n \Gamma(U,\, \mathscr{F}_n) = \Gamma(U, {}_{X_g}\mathscr{F})$. Moreover this identification is compactible with restrictions to quasi-compact open subschemes U' of U. Since U is quasi-compact, by Proposition 2.3.5, we get

$$(\varinjlim_n \mathscr{F}_n)|_{\mathscr{B}} = ({}_{X_g}\mathscr{F})|_{\mathscr{B}}$$

where $\mathscr{B}$ is the basis of quasi-compact open sets on X. Note that since X is quasi-separated, $\mathscr{B}$ is closed under pairwise intersection. By Proposition 2.1.1, we are done. ☐

2.4.8. Suppose $\mathscr{L}$ and $\mathscr{F}$ are quasi coherent $\mathscr{O}_X$-modules with $\mathscr{L}$ invertible, where X is a quasi-compact and quasi-separated scheme. Let $g \in \Gamma(X,\, \mathscr{L})$. Consider the direct system of $\mathscr{O}_X$-modules $(\mathscr{F}\otimes\mathscr{L}^{\otimes n})_n$, with the transition maps from the m^{th} stage to a later n^{th} stage being multiplication by g^{n-m}. Let X_g be the locus of points $x \in X$ such that $g_x \notin \mathfrak{m}_x\mathscr{L}$. One can show that $\varinjlim_n \mathscr{F}\otimes\mathscr{L}^{\otimes n} = {}_{X_g}\mathscr{F}$. The strategy for doing this (we are painting with broad strokes, but the details are easy to fill) is as follows. First note that $\mathscr{L}|_{X_g}$ is canonically isomorphic to $\mathscr{O}_X|_{X_g} = \mathscr{O}_{X_g}$ via the map $g \mapsto 1$ on every open set. Therefore $\mathscr{F}\otimes\mathscr{L}^{\otimes n}|_{X_g} = \mathscr{F}|_{X_g}$, where we are using equality since the identification is canonical. If $j\colon X_g \to X$ is the inclusion map, then we have a natural map $\mathscr{F}\otimes\mathscr{L}^{\otimes n} \to j_*(\mathscr{F}|_{X_g}) = {}_{X_g}\mathscr{F}$ given by the composite

$$\Gamma(U,\, \mathscr{F}\otimes\mathscr{L}^{\otimes n}) \longrightarrow \Gamma(U\cap X_g,\, \mathscr{F}\otimes\mathscr{L}^{\otimes n}) \xrightarrow{\ \sim\ } \Gamma(U\cap X_g,\, \mathscr{F}) = \Gamma(U, {}_{X_g}\mathscr{F})$$

as U varies over open subsets of X. The first map is restriction, and the second is induced by the inverse of "multiplication by g^n", since on X_g, multiplication by g^n is an isomorphism. These maps are compatible with the transition functions on the direct system $(\mathscr{F}\otimes\mathscr{L}^{\otimes n})_n$ and hence we have a map $\varinjlim_n \mathscr{F}\otimes\mathscr{L}^{\otimes n} \to {}_{X_g}\mathscr{F}$. Now modify the strategy in the proof of Lemma 2.4.7 as follows: Take a quasi-compact open subscheme U, cover it with a finite number of affine open subschemes *on each of which $\mathscr{L}$ is trivial,* and cover the pairwise intersection of the members of this cover by a finite number of affine opens. Use analogues of the exact sequences $(\flat)_n$. This gives another proof of Theorem 3.1.1 of Lecture 15.

2.5. **Direct images again.** Theorem 1.2.1 is true for f quasi-compact and quasi-separated. The usual references are either EGA or the Stacks Project [SP]. However, Kempf has a more illuminating proof in [K]. We sketch that. The details are not that difficult to fill by oneself.

Proposition 2.5.1. *Let $f\colon X \to Y$ be a quasi-compact quasi-separated map of schemes and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. Then $f_*\mathscr{F}$ is a quasi-coherent $\mathscr{O}_Y$-module.*

Proof. We remind the reader that for an open set U of X, with $j^U : U \to X$ denoting the inclusion map, ${}_U\mathscr{F}$ is the $\mathscr{O}_X$-module $j^U_*(\mathscr{F}|_U)$, i.e. ${}_U\mathscr{F}(V) = \mathscr{F}(U\cap V)$. It is clear that

$$(\dagger) \qquad f_*({}_U\mathscr{F}) = (f|_U)_*\mathscr{F}.$$

Without loss of generality, we may assume Y is affine. Then X is quasi-compact. Let $\{U_i\}$ be a finite cover of X by affine opens. Since $U_i \cap U_j$ is quasi-compact by Corollary 2.3.3, we have a finite cover $\{V_{ijk}\}_k$ of $U_i \cap U_j$ by affine open subschemes. It follows that we have an exact sequence of $\mathscr{O}_X$-modules

$$(\#) \qquad 0 \longrightarrow \mathscr{F} \longrightarrow \bigoplus_i ({}_{U_i}\mathscr{F}) \longrightarrow \bigoplus_{ijk} ({}_{V_{ijk}}\mathscr{F}).$$

Since f_* is left exact, applying f_* to $(\#)$ and using $(\dagger)$ we get an exact sequence

$$(f_*(\#)) \qquad 0 \longrightarrow f_*\mathscr{F} \longrightarrow \bigoplus_i (f_{U_i})_*(\mathscr{F}|_{U_i}) \longrightarrow \bigoplus_{ijk} (f_{V_{ijk}})_*(\mathscr{F}|_{V_{ijk}}).$$

Since f_{U_i} and $f_{V_{ijk}}$ are affine maps (being maps between affine schemes), by item **4** of § 3.1.1 of Lecture 13, the $\mathscr{O}_Y$-modules $(f_{U_i})_*(\mathscr{F}|_{U_i})$ and $(f_{V_{ijk}})_*(\mathscr{F}|_{V_{ijk}})$ are quasi-coherent $\mathscr{O}_Y$-modules. It is then obvious that $f_*\mathscr{F}$ is quasi-coherent. □

Here is the generalisation of Theorem 1.2.1 to quasi-compact, quasi-separated morphisms.

Theorem 2.5.2. *Let $f\colon X \to Y$ be a quasi-compact, quasi-separated map of schemes and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module.*

(a) $\mathrm{R}^i f_*\mathscr{F}$ *is quasi-coherent for every* $i \geq 0$.
(b) *If Y is affine, say $Y = \operatorname{Spec} A$, and if for each $i \geq 0$, M^i is the A-module $\mathrm{H}^i(X, \mathscr{F})$, then we have a canonical isomorphism $\mathrm{R}^i f_*\mathscr{F} \xrightarrow{\sim} \widetilde{M^i}$ for every $i \geq 0$*

Proof. First note that $\mathrm{R}^i f_*\mathscr{F}$ is the sheafification of the presheaf $P^i(\mathscr{F})$ given by $V \rightsquigarrow \mathrm{H}^i(f^{-1}V, \mathscr{F})$. To see this, let $T^i(\mathscr{F})$ be this sheafification. Then $T^0(\mathscr{F}) = f_*\mathscr{F}$, since the presheaf $V \rightsquigarrow \mathrm{H}^0(f^{-1}V, \mathscr{F})$ is already the sheaf $f_*\mathscr{F}$. Next note that if $\mathscr{E}$ is an injective sheaf, then $T^i(\mathscr{E}) = 0$ for $i \geq 1$, since $\mathscr{E}|_{f^{-1}(V)}$ is injective too (at the very least, it is flasque, if you don't believe the injective statement). Finally, if $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ is a short exact sequence of sheaves, then we have a long exact sequence

$$\dots \longrightarrow T^{n-1}(\mathcal{C}) \longrightarrow T^n(A) \longrightarrow T^n(\mathcal{B}) \longrightarrow T^n(\mathcal{C}) \longrightarrow T^{n+1}(\mathcal{A}) \longrightarrow \dots$$

From the above, it is well known, and easy to prove, that then $T^i \xrightarrow{\sim} \mathrm{R}^i f_*$ for all $i \geq 0$.

Now, without loss of generality, we may assume $Y = \operatorname{Spec} A$. Let $\mathscr{B}$ be the standard basis of the topology on Y, i.e. $\mathscr{B} = \{D_A(g) \mid g \in A\}$. Let $\phi\colon A = \Gamma(Y, \mathscr{O}_Y) \to \Gamma(Y, f_*\mathscr{O}_X) = \Gamma(X, \mathscr{O}_X)$ be the map induced by $f^\#\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$. Let $g \in A$. Then

$$f^{-1}(D(g)) = X_{\phi(g)}.$$

Fix $i \geq 0$ and let (M_n) be the direct system with $M_n = M^i$ for all $n \geq 0$ and whose transition maps are multiplication by g^{n-m}. We point out that the endomorphism of M^i given by multiplication by g is the same as the endomorphism on $\mathrm{H}^i(X, \mathscr{F})$ induced by multiplication by $\phi(g)$ on $\mathscr{F}$. Let $(\mathscr{F}_n)$ be the direct system of $\mathscr{O}_X$-modules with $\mathscr{F}_n = \mathscr{F}$ for every n and such that the transition from $\mathscr{F}_m$ to $\mathscr{F}_n$,

$m \leq n$, is given by multiplication by $\phi(g)^{n-m}$. We have the following sequence of isomorphisms

$$\begin{aligned} M_g^i = \varinjlim_n M_n &= \varinjlim_n \mathrm{H}^i(X,\, \mathscr{F}_n) \\ &\xrightarrow{\sim} \mathrm{H}^i(X,\, \varinjlim_n \mathscr{F}_n) \\ &= \mathrm{H}^i(X,\, {}_{X_{\phi(g)}}\mathscr{F}) \end{aligned}$$

We are using Theorem 2.4.5 for the isomorphism in the second row, and Lemma 2.4.7 for the last identification.

We have a cartesian diagram

$$\begin{array}{ccc} X_{\phi(g)} & \hookrightarrow & X \\ {\scriptstyle f|_{X_{\phi(g)}}}\downarrow & \square & \downarrow{\scriptstyle f} \\ D(g) & \hookrightarrow & Y \end{array}$$

Since the inclusion $D(g) \to Y$ is an affine map, it is clear that the inclusion $j\colon X_{\phi(g)} \to X$ is an affine map. Now ${}_{X_{\phi(g)}}\mathscr{F} = j_*(\mathscr{F}|_{X_{\phi(g)}})$. By part (c) of Theorem 3.1.2 of Lecture 13 we see that $\mathrm{H}^i(X,\, j_*(\mathscr{F}|_{X_{\phi(g)}})) \xrightarrow{\sim} \mathrm{H}^i(X_{\phi(g)},\, \mathscr{F}|_{X_{\phi(g)}})$ whence

$$M_g^i \xrightarrow{\sim} \mathrm{H}^i(X_{\phi(g)},\, \mathscr{F}|_{X_{\phi(g)}}) = \mathrm{H}^i(f^{-1}(D(g)),\, \mathscr{F}).$$

Thus $P^i(\mathscr{F})(D(g)) = \widetilde{M^i}(D(g))$, where $P^i(\mathscr{F})$ is the presheaf defined in the first sentence of this proof. In other words $P^i(\mathscr{F})|_{\mathscr{B}} = \widetilde{M}|_{\mathscr{B}}$. It follows from Proposition 2.1.1 that $\widetilde{M} = (P^i(\mathscr{F}))^+ = \mathrm{R}^i f_*(\mathscr{F})$. This proves (a) and (b) simultaneously. $\square$

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