DIRECT IMAGES OF QUASI-COHERENT SHEAVES - BASIC

The symbol $\mathscr{A}b$ will denote the category of abelian groups and $\mathbb{S}\mathrm{ch}$ the category of schemes. If X is a topological space, $\mathscr{P}sh_X$ and $\mathscr{S}h_X$ denote the category of presheaves and the category of sheaves respectively on X. By a ring we mean a commutative ring with identity. For a ring A, Mod_A denotes the category of A-modules. For a sheaf of rings $\mathscr{A}$ on a topological space, $\mathrm{Mod}_{\mathscr{A}}$ will denote the category of $\mathscr{A}$-modules.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. **Separated and quasi-compact maps**

For any continuous map $f\colon X \to Y$ of topological spaces, the right derived functors $\mathrm{R}^i f_*\colon \mathscr{S}h_X \to \mathscr{S}h_Y$ are called the *higher direct images* of f. This includes the case $i = 0$, i.e. we regard $f_* = \mathrm{R}^0 f_*$ also as a higher direct image. Let me remind you that if $\mathscr{F} \in \mathscr{S}h_X$, then $\mathrm{R}^i f_*\mathscr{F} = \mathrm{H}^i(f_*\mathscr{I}^\bullet)$ where $\mathscr{I}^\bullet$ is a bounded below injective resolution of $\mathscr{F}$.

1.1. **Higher direct images of flasque sheaves.** Let $f\colon X \to Y$ be as above. If $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequence of flasque sheaves on X, it is clear (by testing on open subsets of Y) that $0 \to f_*\mathscr{F} \to f_*\mathscr{G} \to f_*\mathscr{H} \to 0$ is exact. Since injectives are flasque, it then follows (from the formula for $\mathrm{R}^i f_*$ given above and using classical injective resolutions), that $\mathrm{R}^i f_*\mathscr{F} = 0$ for $i \geq 1$ if $\mathscr{F}$ is flasque. Thus flasque sheaves are f_*-acyclic, and hence can be used to compute higher direct images.

If X and Y are ringed spaces, and $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_X}$, then it is easy to see that $\mathrm{R}^i f_*\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_Y}$. Indeed,we can take a bounded below resolution $\mathscr{F} \to \mathscr{I}^\bullet$ by injectives in $\mathrm{Mod}_{\mathscr{O}_X}$, and this will be a flasque resolution. The rest is straightforward, since this observation implies that $\mathrm{R}^i f_*\mathscr{F} \xrightarrow{\sim} \mathrm{H}^i(f_*\mathscr{I}^\bullet)$.

1.2. **Higher direct images of quasi-coherent sheaves.** Recall that a map of schemes $f\colon X \to Y$ is *quasi-compact* if $f^{-1}(V)$ is quasi-compact for every quasi-compact open subset V of Y.

Theorem 1.2.1. *Let $f\colon X \to Y$ be quasi-compact and separated map and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module.*

(a) $\mathrm{R}^i f_*\mathscr{F}$ *is quasi-coherent for every* $i \geq 0$.
(b) *If Y is affine, say $Y = \mathrm{Spec}\, A$, and if for each $i \geq 0$, M_i is the A-module $\mathrm{H}^i(X, \mathscr{F})$, then we have a canonical isomorphism $\mathrm{R}^i f_*\mathscr{F} \xrightarrow{\sim} \widetilde{M_i}$ for every $i \geq 0$*

Proof. The basic idea is straightforward. Part (a) is local on the base, and so we assume, without loss of generality, that $Y = \mathrm{Spec}\, A$. In slightly greater detail, the restriction of a flasque sheaf to an open subset is also flasque, whence higher direct images behave well with respect to restrictions to open subsets of the base, i.e.

$\mathrm{R}^i(f_V)_*(\mathscr{F}|_{f^{-1}(V)}) = \mathrm{R}^i f_*\mathscr{F}|_V$ for every open set V in Y, where $f_V\colon f^{-1}(V) \to V$ is the obvious map induced by f.[1]

We begin with two observations. Since Y is affine, it is quasi-compact and separated. Therefore, since f is quasi-compact and separated, X is quasi-compact and separated. Since X is quasi-compact it has a finite cover $\mathfrak{U} = \{U_0, \ldots, U_d\}$ by affine open subschemes. With $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ as usual denoting the sheaf Čech complex, we will show

- **(i)** The standard resolution $\mathscr{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ of $\mathscr{F}$ is an f_*-acyclic resolution of $\mathscr{F}$, so that $\mathrm{R}^i f_*\mathscr{F} \cong \mathrm{H}^i(\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}))$.
- **(ii)** The $\mathscr{O}_Y$-modules $f_*\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$, $0 \le p \le d$, are all quasi-coherent. This would prove that $\mathrm{H}^i(\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F}))$, and hence by **(i)**, $\mathrm{R}^i f_*\mathscr{F}$ is quasi-coherent.

To that end we introduce some notations. For $0 \le p \le d$, we write $\boldsymbol{i}$ for $(i_0, \ldots, i_p) \in \mathbf{Z}^p$, with $0 \le i_0 < \ldots i_p \le d$, and for such $\boldsymbol{i}$, we set $U_{\boldsymbol{i}} = U_{i_0} \cap \cdots \cap U_{i+p}$. Since X is separated , by Proposition 2.1.4 of Lecture 13, $U_{\boldsymbol{i}}$ is an affine open set. Let $j^{\boldsymbol{i}}\colon U_{\boldsymbol{i}} \to X$ be the inclusion map. Then

$$(*) \qquad \mathscr{C}^p(\mathfrak{U}, \mathscr{F}) = \bigoplus_{0 \le i_0 < \cdots < i_p \le d} j^{\boldsymbol{i}}_*(\mathscr{F}|_{U_{\boldsymbol{i}}}).$$

Now $f_*j^{\boldsymbol{i}}_*(\mathscr{F}|_{U_{\boldsymbol{i}}}) = (f \circ j^{\boldsymbol{i}})_*(\mathscr{F}|_{U_{\boldsymbol{i}}})$, and $f \circ j^{\boldsymbol{i}}\colon U_{\boldsymbol{i}} \to Y$ is an affine map, being a map between affine schemes. It follows that $(f \circ j^{\boldsymbol{i}})_*(\mathscr{F}|_{U_{\boldsymbol{i}}})$ is quasi-coherent by item **4** of § 3.1.1 of Lecture 13. Thus $f_*\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is quasi-coherent. This proves **(ii)**.

We now prove **(i)**. If $j\colon U \to X$ is an open immersion, with U affine, then we have seen earlier that in this case j is affine, since $j^{-1}(V) = U \cap V$ is affine for every affine open subscheme V of the separated scheme X (see Proposition 2.1.4 of Lecture 13). Let $\mathscr{G}$ be quasi-coherent on U. Since j is affine, $j_*\mathscr{G}$ is quasi-coherent on X (see item **4** of § 3.1.1 of Lecture 13). Moreover, from Theorem 3.1.2 of Lecture 13 we see that $j_*\colon \mathrm{Qcoh}(U) \to \mathrm{Qcoh}(X)$ is exact and $\mathrm{R}^n j_*\mathscr{G} = 0$ for $n \ge 1$ (since, by our choice, $\mathscr{G} \in \mathrm{Qcoh}(U)$). Now suppose $\mathscr{G} \to \mathscr{I}^\bullet$ is a classical resolution of $\mathscr{G}$ by injective objects in $\mathrm{Mod}_{\mathscr{O}_U}$. From the above observations, $j_*\mathscr{I}^\bullet$ is a flasque resolution of $j_*\mathscr{G}$. It follows that $\mathrm{R}^n f_*(j_*\mathscr{G}) \cong \mathrm{H}^n(f_*j_*\mathscr{I}^\bullet) = \mathrm{H}^n((f \circ j)_*\mathscr{I}^\bullet) = \mathrm{R}^n(f \circ j) * \mathscr{G}$. Now $f \circ g\colon U \to Y$ is a map of affine schemes, and hence is an affine. One more application of Theorem 3.1.2 of Lecture 13 gives that $\mathrm{R}^n(f \circ g)_*\mathscr{G} = 0$ for $n \ge 1$, i.e., from the string of isomorphisms we just wrote, $\mathrm{R}^n f_*(j_*\mathscr{G}) = 0$ for $n \ge 1$. In other words $j_*\mathscr{G}$ is f_*-acyclic. In particular, the summand $j^{\boldsymbol{i}}_*(\mathscr{F}|_{U_{\boldsymbol{i}}})$ in the direct sum decomposition of $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ in $(*)$ is f_*-acyclic. This proves **(i)**. We have therefore proved (a) of the Theorem.

For (b) we use the cover $\mathfrak{U}$ above and continue to use the notations we have introduced above. Since Y is affine, $\Gamma(Y, -)\colon \mathrm{Qcoh}(Y) \to \mathrm{Mod}_A$ and $\widetilde{(-)}\colon \mathrm{Mod}_A \to \mathrm{Qcoh}(Y)$ are pseudo-inverses to each other, and both are exact functors. It is therefore enough to show that $\Gamma(Y, \mathrm{R}^i f_*\mathscr{F}) \xrightarrow{\sim} M_i$. This follows from the following

[1] The statement about the higher direct image being well-behaved with respect to restrictions to open subsets on the base is a topoliogical statement and not a scheme-theoretic one; so just for that statement X and Y are topoligical spaces and f a continuous map.

sequence of isomorphisms (explanations after the displayed sequence of isomorphisms):

$$\begin{aligned}\Gamma(Y,\, \mathrm{R}^i f_*\mathscr{F}) &\xrightarrow{\sim} \Gamma(Y,\, \mathrm{H}^i(\mathscr{C}^\bullet(\mathfrak{U},\, \mathscr{F})))\\ &\xrightarrow{\sim} \mathrm{H}^i(\Gamma(Y,\, \mathscr{C}^\bullet(\mathfrak{U},\, \mathscr{F})))\\ &=\!=\; \mathrm{H}^i(C^\bullet(\mathfrak{U},\, \mathscr{F}))\\ &\xrightarrow{\sim} \mathrm{H}^i(X,\, \mathscr{F}).\end{aligned}$$

The fist isomorphism is from the statement in **(i)**. The second isomorphism follows from the fact that $\Gamma(Y, -)$ is exact, whence commutes with cohomology. The last isomorphism follows from Theorem 3.2.4 of of Lecture 13, since X is separated. □

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