ČECH COMPLEX, TENSOR PRODUCT OF COMPLEXES, $\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)$ AND THE KOSZUL COMPLEX

1. The Čech complex

Here is one version of "the" Čech complex. Technically, this is the *ordered* Čech complex. This is the version given for example in Hartshorne's *Algebraic Geometry.* This will be the default version we'll use in the course, though we may give other versions, which are obviously quasi-isomorphic to this one. See Section 01FG and Section 01ED of *The Stacks project* if you wish to read more now.

Let X be a topological space and $\mathfrak{U} = (U_i)_{i\in I}$ a covering of X, with the index set I having a fixed well-ordering [1]. Write $U_{i_0\ldots i_p}$ for the intersection $U_{i_0\ldots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. Define for any *presheaf* $\mathscr{P}$ on X

$$C^p(\mathfrak{U},\, \mathscr{P}) := \prod_{i_0<\cdots<i_p} \mathscr{P}(U_{i_0\ldots i_p}).$$

Define the coboundary $C^p(\mathfrak{U},\, \mathscr{P}) \to C^{p+1}(\mathfrak{U},\, \mathscr{P})$ in the usual way, namely via the standard simplicial formula:

$$(d^p(s))_{i_0\ldots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k s_{i_o\ldots\hat{k}\ldots i_{p+1}}|_{U_{i_0\cdots i_{p+1}}}.$$

The i^{th} *Čech cohomology* $\check{\mathrm{H}}^i(\mathfrak{U},\, \mathscr{P})$ of $\mathscr{P}$ with respect to $\mathfrak{U}$ is the i^{th} cohomology of the complex $C^p(\mathfrak{U},\, \mathscr{P})$:

$$\check{\mathrm{H}}^i(\mathfrak{U},\, \mathscr{P}) := \mathrm{H}^i(C^\bullet(\mathfrak{U},\, \mathscr{P})).$$

2. The tensor product of two complexes and $\mathrm{Hom}^\bullet(A^\bullet,\, B^\bullet)$

2.1. Tensor product of complexes.

Suppose A is a commutative ring (or more generally a sheaf of rings over a topological space). If $R^\bullet$ and $S^\bullet$ are complexes of A-modules, then we define the tensor product $R^\bullet \otimes_A S^\bullet$ of $R^\bullet$ and $S^\bullet$ as the complex whose n^{th} term is

$$[R^\bullet \otimes_A S^\bullet]^n = \bigoplus_{p+q=n} R^p \otimes_A S^q$$

and whose coboundary is (with $r^p \in R^p$ and $s^q \in S^q$)

$$\partial^{p+q}(r^p \otimes s^q) = \partial^p(r^p) \otimes s^q + (-1)^p r^p \otimes \partial^q(s^q).$$

[1] By the Axiom of Choice, there is at least one well-ordering on I and hence at least one total order.

2.2. **The complex $\mathbf{Hom}^\bullet(A^\bullet,\, B^\bullet)$.** Now suppose $\mathscr{A}$ is an abelian category, and $A^\bullet$, $B^\bullet$ are complexes. We define a complex $\mathrm{Hom}^\bullet_{\mathscr{A}}(A^\bullet,\, B^\bullet)$ as follows (and to lighten notation, we drop $\mathscr{A}$ from the subscript for Hom):

$$\mathrm{Hom}^n(A^\bullet,\, B^\bullet) = \prod_{j\in\mathbb{Z}} \mathrm{Hom}(A^j,\, B^{j+n}).$$

The coboundary $d^n\colon \mathrm{Hom}^n(A^\bullet,\, B^\bullet) \to \mathrm{Hom}^{n+1}(A^\bullet,\, B^\bullet)$ takes $f = (f^j)_{j\in\mathbb{Z}}$ in $\prod_{j\in\mathbb{Z}} \mathrm{Hom}(A^j,\, B^{j+n}) = \mathrm{Hom}^n(A^\bullet,\, B^\bullet)$ to $d^n(f) \in \prod_{j\in\mathbb{Z}} \mathrm{Hom}(A^j,\, B^{j+n+1}) = \mathrm{Hom}^{n+1}(A^\bullet,\, B^\bullet)$ and the formula is:

$$d^n(f) = (\partial^{n+j}_{B^\bullet} \circ f^j + (-1)^{n+1} f^{j+1} \circ \partial^j_{A^\bullet})_{j\in\mathbb{Z}}.$$

This formula is standard (see for example B. Iversen's *Cohomology of Sheaves* or J. Lipman and M.Hashimoto's rather advanced *Foundations of Grothendieck Duality for Diagrams of Schemes*). Unfortunately a classic in the subject—Robin Hartshorne's *Residues and Duality*—has a different convention and it differs from the above by a factor of $(-1)^{n+1}$.

3. Koszul Complexes

3.1. There are two related Koszul complexes. The *homology Kozsul complex* and the *cohomology Koszul complex.* Let A be a commutative ring and t an element of A and M an A-module. The cohomology Koszul complex of M with respect to t is the complex

$$K^\bullet(t, M): \quad 0 \to M \xrightarrow{t} M \to 0 \tag{3.1.1}$$

with the left M in the 0^{th} spot and whence the right M in the first spot. Note that

$$K^\bullet(t,\, M) = K^\bullet(t,\, A) \otimes_A M.$$

If $\mu, \nu \in \mathbf{N}$ are such that $\mu \leq n$, then for $t \in A$, then note that we have a map of complexes $\theta_{\mu,\nu}(t)\colon K^\bullet(t^\mu,\, M) \to K^\bullet(t^\nu,\, M)$ which is the identity on $K^0(t^\mu, M) = K^0(t^\nu,\, M) = M$ and is multiplication by $t^{\nu-\mu}$ on $K^1(t^\mu, M) = K^1(t^\nu,\, M) = M$. It is easy to see that $(K^\bullet(t^\nu,\, M))_\nu$ is then a *direct system of complexes.*

If $\mathbf{t} = (t_1, \ldots, t_n)$ is a sequence of elements in A, then the *Koszul cohomology complex of M with respect to* $\mathbf{t}$ is

$$K^\bullet(\mathbf{t},\, M) := K^\bullet(t_1,\, A) \otimes_A \ldots \otimes_A K^\bullet(t_n,\, A) \otimes_A M.$$

This complex lives in degrees $0,\, 1, \ldots, n$.[2]

If $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)$ are two elements in $\mathbf{N}^n$ with $\mu_i \leq \nu_i$, $i = 1, \ldots, n$, then we have a map of complexes $\theta_{\boldsymbol{\mu\nu}}\colon K^\bullet(\mathbf{t}^{\boldsymbol{\mu}},\, M) \to K^\bullet(\mathbf{t}^{\boldsymbol{\nu}},\, M)$ given by tensoring the maps $\theta_{\mu_i,\nu_i}(t_i)\colon K^\bullet(t_i^{\mu_i}\, A) \to \mathbf{K}^\bullet(t_i^{\nu_i}\, A)$ defined above, and then tensoring the result with M. We thus have a direct system of complexes indexed by the directed set $\mathbf{N}^n$ with the obvious partial order $\prec$ (so that $\boldsymbol{\mu} \prec \boldsymbol{\nu}$ for $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ as above):

$$\Big(K^\bullet(\mathbf{t}^{\boldsymbol{\nu}},\, M), \theta_{\boldsymbol{\mu\nu}}\Big). \tag{3.1.2}$$

The homology Koszul complexes are identical expect for the grading involved. Thought of as a co-chain complex it lives in degrees $-n, -n+1, \ldots, 0$ (and hence

[2] This is an n-fold tensor product of complexes, and rightly should be arranged in an n-dimensional grid. There is an associative law for tensor product of complexes, and in any case, it is enough to think of $A_1^\bullet \otimes \ldots \otimes A_n^\bullet$ as $((\ldots((A_1^\bullet \otimes A_2^\bullet) \otimes A_3^\bullet) \ldots) \otimes A_{n-1}^\bullet) \otimes A_n^\bullet$.

as a chain complex, i.e. as a homology complex, it lives in degrees $0, \dots, n$). Recall that one can turn a cohomology complex $C^\bullet$ into a homology complex $C_\bullet$ by setting $C_i = C^{-i}$. Then the Koszul homology complex of M with respect to $\mathbf{t}$ is obtained by shifting $K^\bullet(\mathbf{t}, M)$ n-places to the left (no change of sign) and then regarding it as a homology complex. As a chain complex (=homology complex) it is invoked by the symbol $K_\bullet(\mathbf{t}, M)$. It is easy to see that

$$K_\bullet(\mathbf{t}, M) = K_\bullet(t_1, A) \otimes_A \dots \otimes_A K_\bullet(t_n, A) \otimes_A M. \tag{3.1.3}$$

An interesting fact is that $K^\bullet(\mathbf{t}, M) = \mathrm{Hom}_A(K_\bullet(\mathbf{t}, A), M)$ where the complex on the right is the naive complex (without the fancy sign conventions of the previous sections).

There is another way to define $K^\bullet(\mathbf{t}, M)$. Let For each $p = 0, \dots, d$ let $K_p^{\boldsymbol{e}}(\mathbf{t})$ be the free A-module of rank $\binom{d}{p}$ on the free generators $\{e_{i_1\cdots i_p} \mid 1 \le i_1 < \cdots < i_p \le d\}$, i.e.,

$$K_p^{\boldsymbol{e}}(\mathbf{t}) := \bigoplus_{1 \le i_1 < \cdots < i_p \le d} A e_{i_1 \cdots i_p}$$

with the understanding that $K_0^{\boldsymbol{e}}(\mathbf{t}) = A$. Set

$$K^p(\mathbf{t}, M) = \mathrm{Hom}_A(K_p^{\boldsymbol{e}}(\mathbf{t}), M). \tag{3.1.4}$$

The differentials $d^p\colon K^p(\mathbf{t}, M) \to K^{p+1}(\mathbf{t}, M)$ are defined as follws. When $n = 0$, $K^\bullet(\mathbf{t}, M) = M$ and all differentials are zero. When $n > 0$ the definition for d^p is

$$d^p(\tau)(e_{i_1 \dots e_{i_{p+1}}}) = \sum_{j=1}^{p+1} (-1)^{j+1} t_{i_j} \tau(e_{i_1 \dots \widehat{i_j} \dots i_{p+1}}). \tag{3.1.5}$$

It is easy to see that the two definitions are equivalent.

The most well known result concerning Koszul complexes involves the notion of an M-sequence. So let M and $\mathbf{t}$ be as above. We say $\mathbf{t}$ is an M-sequence if t_1 is a non-zero divisor of M, i.e. the M-endomorphism $x \mapsto tx$ is injective, and for $1 < p \le n$ t_p is a non-zero divisor on $M/\langle t_1, \dots, t_{p-1}\rangle M$. The following lemma is well known (see [M, p.128, Theorem 16.5], and use the fact that $K^\bullet(\mathbf{t}, M)$ is isomorphic the translate of $K_\bullet(t, M)$ by n-units.)

Lemma 3.1.6. *Let $\mathbf{t} = (t_1, \dots, t_n)$ be an M-sequence. Then*

$$\mathrm{H}^p(K^\bullet(\mathbf{t}, M)) = \begin{cases} 0 & \text{when } 0 \le p < n \\ M/\mathbf{t}M & \text{when } p = n. \end{cases}$$

Koszul complexes are intimately related to Čech complexes as your Homework will show.

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