LECTURE 9

Date of Lecture: Oct 21, 2021

The symbol $\mathscr{Ab}$ will denote the category of abelian groups and $\mathbb{S}\mathrm{ch}$ the category of schemes. If X is a topological space, $\mathscr{Psh}_X$ and $\mathscr{Sh}_X$ denote the category of presheaves and the category of sheaves respectively on X. By a ring we mean a commutative ring with identity. The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Ring homomorphisms and scheme maps

As in HW 3 we will often use the notation $\Gamma(U, F)$ for $F(U)$, where F is a presheaf on a topological space X and U an open subset of X.

1.1. The open set $D(f)$.

Let X be a locally ringed space. For $x \in X$, $\mathfrak{m}_x$ will denote the maximal ideal of $\mathscr{O}_{X,x}$ and for any section f of $\mathscr{O}_X$ over an open neighbourhood of x, f_x will denote the image of f in $\mathscr{O}_{X,x}$. In other words f_x is the germ of f at x. Let $f \in \Gamma(X, \mathscr{O}_X)$. Define

$$D(f) = \{x \in X \mid f_x \notin x\}. \tag{1.1.1}$$

From Problem 2 of HW 3 we know that $D(f)$ is open in X. Indeed, if $x \in D(f)$, then according to *loc.cit.* there is an open neighbourhood U of x on which $f|_U$ is invertible, whence $U \subset D(f)$, proving that $D(f)$ is open, and that f is invertible on $D(f)$, since two local inverses have to agree on the intersection of their domains. In fact from this argument, $D(f)$ is the maximal open set U such that $f|_U$ is invertible.

We point out that if X is the spectrum of a ring A then $D(f)$ is precisely what we denoted as X_f in §2.2 of Lecture 2, i.e $D(f) = \mathrm{Spec} A_f$.

In the proposition that follows $\mathfrak{Lrs}$ is the category of locally ringed spaces, and $\mathfrak{Rng}$ the category of rings (always commutative with multiplicative identity).

Theorem 1.1.2. *Let X and Y be locally ringed spaces with Y an affine scheme. Then the map $(\psi, \psi^\#) \mapsto \Gamma(Y, \psi^\#)$ gives us a bijection*

$$\mathrm{Hom}_{\mathfrak{Lrs}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{Rng}}(\Gamma(Y, \mathscr{O}_Y), \Gamma(X, \mathscr{O}_X)).$$

Proof. Let $A = \Gamma(Y, \mathscr{O}_Y)$, $B = (\Gamma(X, \mathscr{O}_X)$,and let $\phi\colon A \to B$ be a ring map. Define a set theoretic map $\psi\colon X \to Y$ as follows. For $x \in X$, let $\mathfrak{m}_x$ be rhe maximal ideal of $\mathscr{O}_X$. Let $\mathfrak{p}$ be the inverse image of $\mathfrak{m}_x$ under the composite map $A \xrightarrow{\phi} B \to \mathscr{O}_{X,x}$ and y the point in Y corresponding to $\mathfrak{p}$. Set $\psi(x) = y$. We claim ψ is continuous. Indeed take a basic open subset of the form $D(f)$ in Y, where $f \in A$. It is clear that $\psi^{-1}(D(f)) = D(\phi(f))$. Thus ψ is continuous. Next we define $\psi^\#\colon \mathscr{O}_Y \to \psi_*\mathscr{O}_X$. Let $f \in A$. Consider the composite

$$A \xrightarrow{\phi} B = \Gamma(X, \mathscr{O}_X) \xrightarrow{\text{restriction}} \Gamma(D(\phi(f)), \mathscr{O}_X).$$

Since $\phi(f)|_{D(\phi(f))}$ is invertible, it follows that the image of f under the above composite is invertible. By the universal property of localisation we therefore get a map of rings $\psi^\#_f\colon A_f \to \Gamma(D(\phi(f)),\, \mathscr{O}_X)$ such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "\phi"] \arrow[d, "\text{localisation}"'] & B \arrow[r, equal] & \Gamma(X,\, \mathscr{O}_X) \arrow[d, "\text{restriction}"] \\
A_f \arrow[rr, "\psi^\#_f"'] & & \Gamma(D(\phi(f)),\, \mathscr{O}_X)
\end{tikzcd}$$

This can be expanded to the commutative diagram

$$\begin{tikzcd}
\Gamma(Y,\, \mathscr{O}_Y) \arrow[r, equal] \arrow[d, "\text{restriction}"'] & A \arrow[r, "\phi"] \arrow[d, "\text{localisation}"] & B \arrow[r, equal] & \Gamma(X,\, \mathscr{O}_X) \arrow[d, "\text{restriction}"] \\
\Gamma(D(f),\, \mathscr{O}_Y) \arrow[r, equal] \arrow[d, equal] & A_f \arrow[rr, "\psi^\#_f"'] & & \Gamma(D(\phi(f)),\, \mathscr{O}_X) \arrow[d, equal] \\
\Gamma(D(f),\, \mathscr{O}_Y) \arrow[rrr, "\psi^\#_f"'] & & & \Gamma(D(f),\, \psi_*\mathscr{O}_X)
\end{tikzcd}$$

One checks that the arrow in the bottom row, as f is allowed to vary in A, gives us a map of $\mathscr{B}$-sheaves for the standard basis $\mathscr{B}$ of $Y = \operatorname{Spec} A$. This gives us a map of sheaves

$$\psi^\#\colon \mathscr{O}_Y \to \psi_*\mathscr{O}_X.$$

The map $\psi^\#$ is clearly a map of sheaves of rings. From the definition of $\psi^\#$ it is clear that $\Gamma(Y,\, \psi^\#) = \phi$. Using Problem 1 of HW 3 one checks that the map $\phi \mapsto (\psi,\, \psi^\#)$ is the inverse of the map $(\psi,\, \psi^\#) \mapsto \Gamma(Y,\, \psi^\#)$. □

Let $\mathscr{Aff}$ be the full subcategory of $\mathbb{S}\mathrm{ch}$ whose objects are affine schemes.

The theorem gives another proof of the following result (as an immediate corollary).

Corollary 1.1.3. *The contravariant functor $X \mapsto \Gamma(X,\, \mathscr{O}_X)$ on affine schemes is an anti-equivalence of categories between $\mathscr{Aff}$ and $\mathfrak{Rng}$.*

A second corollary is the following.

Corollary 1.1.4. *Let X be a scheme, $X_\Gamma = \operatorname{Spec}(\Gamma(X,\, \mathscr{O}_X)$ and $X \to X_\Gamma$ the map arising from the theorem via the identity map on $\Gamma(X,\, \mathscr{O}_X) = \Gamma(X_\Gamma,\, \mathscr{O}_{X_\Gamma})$. If $f\colon X \to Y$ is a map of schemes with Y an affine scheme, then there is a unique map $f_\Gamma\colon X_\Gamma \to Y$ such that the following diagram commutes*

$$\begin{tikzcd}
X \arrow[d] \arrow[dr, "f"] & \\
X_\Gamma \arrow[r, "f_\Gamma"'] & Y
\end{tikzcd}$$

Proof. The theorem tells us that there are bijections

$$\begin{aligned}\mathrm{Hom}_{\mathbb{S}\mathrm{ch}}(X,Y) &\xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{Rng}}(\Gamma(Y,\,\mathscr{O}_Y),\,\Gamma(X,\,\mathscr{O}_X))\\ &= \mathrm{Hom}_{\mathfrak{Rng}}(\Gamma(Y,\,\mathscr{O}_Y),\,\Gamma(X_\Gamma,\,\mathscr{O}_{X_\Gamma}))\\ &\xleftarrow{\sim} \mathrm{Hom}_{\mathbb{S}\mathrm{ch}}(X_\Gamma,\,Y).\end{aligned}$$

Thus there is a map $f_\Gamma\colon X_\Gamma \to Y$ corresponding to f under the above bijection. The bijections are functorial in $Y \in \mathscr{Aff}$. Setting $Y = X_\Gamma$ and using the standard Yoneda argument, one sees that diagram in the statement of this corollary commutes. □

2. $\mathscr{O}_X$-modules and quasi-coherent sheaves on affine schemes

2.1. **$\mathscr{O}_X$-modules and $\mathscr{O}_X$-algebras.** Let X be a ringed space. A sheaf $\mathscr{F}$ on X is called a $\mathscr{O}_X$-module if $\mathscr{F}(U)$ is a $\mathscr{O}_X(U)$-module for every open set U in X and if each an inclusion of open sets $U \subset V$ the restriction map $\mathscr{F}(V) \to \mathscr{F}(U)$ is a $\mathscr{O}_X(V)$-module map. Here $\mathscr{F}(U)$ is regarded as an $\mathscr{O}_X(V)$-module via the ring homomorphism $\mathscr{O}_X(V) \to \mathscr{O}_X(U)$. $\mathscr{O}_X$-modules form a category with morphisms being sheaf maps which over every open set U and $\mathscr{O}_X(U)$-module maps.

A sheaf of rings $\mathscr{A}$ on X is called an $\mathscr{O}_X$-algebra if there is a map of sheaf of rings $\mathscr{O}_X \to \mathscr{A}$. In greater detail, this is a sheaf map such that $\mathscr{O}_X(U) \to \mathscr{A}(U)$ is a ring homomorphism for each open U. Note that if $\mathscr{A}$ is an $\mathscr{O}_X$-algebra then $(X,\,\mathscr{A})$ is also a ringed space, and an $\mathscr{A}$-module is necessarily an $\mathscr{O}_X$-module.

2.2. **Quasi-coherent sheaves on affine schemes.** Let X be an affine scheme and $A = \Gamma(X,\,\mathscr{O}_X)$. Let $\mathscr{B} = \{D(f)\}$ be the standard basis for the topology on X. Suppose M is an A-module. Then the assignment $D(f) \mapsto M_f$ gives a $\mathscr{B}$-presheaf with restriction maps being (further) localisation maps. In fact this pre-sheaf is a $\mathscr{B}$-sheaf as is easily checked and we get a sheaf $\widetilde{M}$ on X. The proof that $D(f) \mapsto M_f$ gives a $\mathscr{B}$-sheaf is, *mutatis mutandis*, the proof we gave earlier in the course that $D(f) \mapsto A_f$ is a $\mathscr{B}$-sheaf. Details are left to you. Since M_f is an A_f-module and this module structure is compatible with further localisations, $\widetilde{M}$ is actually an $\mathscr{O}_X$-module. A *quasi-coherent* sheaf on X is an $\mathscr{O}_X$-module $\mathscr{F}$ such that $\mathscr{F}$ is isomorphic as an $\mathscr{O}_X$-module to $\widetilde{M}$ for some $M \in \mathrm{Mod}_A$. It is clear that quasi-coherent sheaves on affine schemes form a subcategory of the category $\mathrm{Mod}_{\mathscr{O}_X}$ of $\mathscr{O}_X$-modules. In fact it forms a full subcategory of $\mathrm{Mod}_{\mathscr{O}_X}$. There are many notations used in the literature for this subcategory. In these lectures we will use the notation X_{qc} for this subcategory. In the tutorials a different notation was probably used. Note that $\mathscr{F} \mapsto \Gamma(X,\,\mathscr{F})$, $\mathscr{F} \in X_qc$ and $M \mapsto \widetilde{M}$, $M \in \mathrm{Mod}_A$, are pseudo-inverse functors giving an equivalence of categories between X_{qc} and Mod_A.

3. $\mathscr{Sh}_X$ is an abelian category

Throughout this section X is a topological space.

3.1. **Stalks of the sheafification.** Suppose $F \in \mathscr{Psh}_X$ and $\mathscr{F} = F^+$ the sheafification of F. Let $x \in X$, U an open neighbourhood of x and $\sigma \in \mathscr{F}(U)$. Then sigma is a continuous section of $\pi = \pi_F\colon \mathscr{E}(F) \to X$ over U, where $\mathscr{E}(F)$ is the étale space associated to F and π is the natural projection. In other words $\sigma\colon U \to \mathscr{E}(F)$ is a continuous map such that $\pi\circ\sigma = \mathbf{1}_U$. We claim that there is an open neighbourhood V of x in U such that there exists $s \in F(V)$ such that $\sigma(y) = s_y$ for $y \in V$, where s_y denotes the image (i.e. the germ) of s in F_y. To see this suppose $\sigma(x) = \xi$.

Then there exists an open set U', which may be assumed to lie in U, with $x \in U'$ and $s' \in F(U')$ such that $s'_x = \xi$. Let $W(s', U)$ be the open subset $\{s'_y \mid y \in U'\}$ of $\mathscr{E}(F)$ (see (2.1.2) of Lecture 1). Note that $\xi \in W(s', U')$. Let $V = \sigma^{-1}(W(s', U'))$. Then $x \in V \subset U' \subset U$. Set $s = s'|_V \in \mathscr{F}(V)$. Then $\sigma(y) = s_y$ for $y \in V$, as claimed.

Allowing σ to vary in $\mathscr{F}(U)$, the association $\sigma \mapsto \sigma(x)$ gives us an map of abelian groups $\mathscr{F}(U) \to F_x$. This behaves well with restrictions to smaller open neighbourhoods of x, whence by the universal property of direct limits we get a well defined map

$$\mathscr{F}_x \to F_x. \tag{3.1.1}$$

It is clear this is map is surjective, since $\xi \in F_x$ is represented by some $s \in F(U)$ for some neighbourhood U of x, and if $\sigma \in \mathscr{F}(U)$ is the image of s under $\theta_F(U)$, then σ_x maps to ξ under (3.1.1). We now show that (3.1.1) is injective. To that end, suppose $\zeta \in \mathscr{F}_x$ maps to zero under (3.1.1). Now ζ is the germ of some section $\sigma \in \mathscr{F}(U)$ where U is an open neighbourhood of x. From the description of (3.1.1), $\sigma(x) = 0$. From the discussion in the first paragraph of this subsection, by shrinking U around x if necessary, we may assume there exists $s \in F(U)$ such that $\sigma(y) = s_y$ for $y \in U$. Since $s_x = \sigma(x) = 0$, there is an open neighbourhood V of x in U such that $s|_V = 0$. Then $s_y = 0$ for $y \in V$, whence $\sigma|_V = 0$. It follows that the germ of σ at x is zero, i.e. $\zeta = 0$. This proves that (3.1.1) is an isomorphism. There is an easy description of the inverse of (3.1.1). Let $\theta = \theta_F \colon F \to \mathscr{F}$ be the sheafification map in (2.2.2) of Lecture 1. Then we have a map of stalks $\theta_x \colon F_x \to \mathscr{F}_x$. One checks easily that θ_x is the inverse of (3.1.1). From now on we make the identification

$$F_x = \mathscr{F}_x \tag{3.1.2}$$

for every $x \in X$.

One important consequence of (3.1.2) is the following:

Theorem 3.1.3. *Let $\varphi \colon F \to G$ be a morphism in $\mathscr{P}sh_X$. The map $\varphi^+ \colon F^+ \to G^+$ is an isomorphism if and only if $\varphi^+_x \colon F_x \to G_x$ is an isomorphism for every $x \in X$. In particular, a map of sheaves $\psi \colon \mathscr{F} \to \mathscr{G}$ is an isomorphism if and only if ψ_x is an isomorphism for every $x \in X$.*

Proof. Consider the map on étale spaces $\mathscr{E}(\varphi) \colon \mathscr{E}(F) \to \mathscr{E}(G)$. It is easy to see that it is a continuous map. Since a basic open set $W(s, U) \subset \mathscr{E}(F)$ maps to $W(\varphi(s), U)$ under $\mathscr{E}(\varphi)$, it is clear that $\mathscr{E}(\varphi)$ is also an open map. Thus if it is bijective it is a homeomorphism. Since $\varphi^+(U)$, for an open set U in X, is the map $\sigma \mapsto \mathscr{E}(\varphi) \circ \sigma$, we are done. □

3.2. **Kernels, cokernels, images, and coimages.** Let $\varphi \colon F \to G$ be a map of pre-sheaves on X. For an open set U in X, let $K_p(U)$ and $C_p(U)$ be defined by

$$K_p(U) = \ker \varphi, \quad C_p(U) = \operatorname{coker} \varphi \tag{3.2.1}$$

If $V \subset U$, then the natural map $K_p(U) \to F(V)$ given by the composite $K_p(U) \hookrightarrow F(U) \xrightarrow{\rho^U_V} F(V)$ must factor through $K_p(V)$ by the universal property of the kernel of $F(V) \to G(V)$ since the composite $K_p(U) \to F(V) \xrightarrow{\varphi(U)} G(U)$ is zero (why?). It is easy to check (again use universal properties of kernels) with this map as the restriction map, that $U \mapsto K_p(U)$ is a presheaf on X and by construction a sub-presheaf of F. Similarly, C_p is a presheaf on X (use, repeatedly, the universal property of cokernels), and we have a map of presehaves $\pi_p\colon G \to C_p$ which for every open U gives a surjective map of groups. In particular it is an epimorphism in the category of presheaves (i.e. it can always be cancelled from the right: if $\alpha\circ\pi_p = \beta\circ\pi_p$ then $\alpha = \beta$). Similarly, $K_p \to F$ is a monomorphism, i.e. it can be cancelled from the left.

It is easy to see that the inclusion $K_p \hookrightarrow F$ has the universal property of kernels, namely, if $\psi\colon H \to F$ is a map in $\mathscr{P}sh$ such that $\varphi\circ\psi = 0$ then ψ factors uniquely as $H \to K_p \hookrightarrow F$. Similarly $\pi_p\colon G \to C_p$ has the universal property of cokernels, namely, if $\psi\colon G \to H$ is a map such that $\psi\circ\varphi = 0$, then there is a unique map $C_p \to H$ such that ψ factors uniquely as $G \xrightarrow{\pi_p} C_p \to H$.

The presheaf K_p is called the *presheaf kernel* of φ. More precisely the monomorphism $K_p \hookrightarrow F$ is called the presheaf cokernel of φ. This kernel will often be denoted $\ker_p(\varphi)$ or simply $\ker(\varphi)$. Similarly C_p, or more precisely π_p, is called the *presheaf cokernel of* φ and will often be denoted $\operatorname{coker}_p(\varphi)$.

Thus kernels and cokernels exist in $\mathscr{P}sh_X$. This raises the question: Is $\mathscr{P}sh_X$ an abelian category?[1] It is, and we formally state the result below, omitting the straightforward proof.

Lemma 3.2.2. *$\mathscr{P}sh_X$ is an abelian category.*

Next suppose we have a map of sheaves $\varphi\colon \mathscr{F} \to \mathcal{G}$. It is easy to check that $\ker_p\varphi$ is a sheaf, since the glueing conditions defining a sheaf structure are inherited by $\ker_p\varphi$ from the sheaf $\mathscr{F}$. Moreover $\ker_p(\varphi)$ has the universal property of kernels in the category $\mathscr{S}h_X$. We can therefore drop the suffix "p" and simply write $\ker\varphi$. However, $C_p = \operatorname{coker}_p(\varphi)$ need not be a sheaf (even though $\mathscr{F}$ and $\mathscr{G}$ are sheaves!). We set $C_s = C_p^+$. The map $\pi_s\colon \mathscr{G} \to C_s$ which is the composite $\mathscr{G} \xrightarrow{\pi_p} C_p \to C_s$, has the universal property of cokernels in the category $\mathscr{S}h_X$. Indeed if $\psi\colon \mathscr{G} \to \mathscr{H}$ is a map of sheaves such that $\psi\circ\varphi = 0$ we have,since C_p has universal property of cokernels in $\mathscr{P}sh_X$, a presheaf map $\psi_p\colon C_p \to \mathscr{H}$ such that $\psi = \psi_p\circ\pi_p$. Now the universal property of sheafifications, we see that we have unique map $\psi_s\colon C_s \to \mathscr{H}$ such that ψ_p is the composite $C_p \to C_s \xrightarrow{\psi_s} \mathscr{H}$. It follows that $\psi = \psi_s\circ\pi_s$, proving that $\pi_s\colon \mathscr{G} \to C_s$ is the cokernel of φ in $\mathscr{S}h_X$. We denote it $\operatorname{coker}_s(\varphi)$.

As is standard for additive categories, we define the *coimage* **im** (φ) and the *image* **im** (φ) of φ by the formulas:

$$\mathbf{coim}\,(\varphi) = \operatorname{coker}\,(\ker(\varphi)) \quad\text{and}\quad \mathbf{im}\,(\varphi) = \ker(\operatorname{coker}\,(\varphi)) \tag{3.2.3}$$

The universal property of kernels and cokernels give us a canonical map (and this is true in any additive category) **coim** $(\varphi) \to$ **im** (φ) such that the diagram

$$\begin{tikzcd} \mathscr{F} \arrow[r, "\varphi"] \arrow[d] & \mathscr{G} \\ \mathbf{coim}\,(\varphi) \arrow[r] & \mathbf{im}\,(\varphi) \arrow[u] \end{tikzcd} \tag{3.2.4}$$

commutes. In greater detail,[2] since the composite $\mathscr{F} \xrightarrow{\varphi} \mathscr{G} \to \operatorname{coker}\varphi$ is zero, so by the universal property of kernels, we have a natural map $\mathscr{F} \to \mathbf{im}\,(\varphi)$ such that φ is the composite $\mathscr{F} \to \mathbf{im}\,(\varphi) \to \mathscr{G}$. Since $\mathbf{im}\,(\varphi) \to \mathscr{G}$ is a monomorphism

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[1] See https://www.cmi.ac.in/~pramath/GradAlgI_2015/Notes/notes4.pdf for definitions.

[2] Skip these details on a first reading if you are uncomfortable with the abstractions.

(being a kernel), it follows that the composite $\ker(\varphi) \hookrightarrow \mathscr{F} \to \mathbf{im}\,(\varphi)$ is zero since this composite followed by the monomorphism $\mathbf{im}\,(\varphi) \to \mathscr{G}$ is zero, and we can cancel monomorphisms from the left. By the universal property of cokernels, we then have a unique map $\mathbf{coim}\,(\varphi) \to \mathbf{im}\,(\varphi)$ such that $\mathscr{F} \to \mathbf{im}\,(\varphi)$ factors as $\mathscr{F} \to \mathbf{coim}\,(\varphi) \to \mathbf{im}\,(\varphi)$.

Using (3.1.2) we see that the stalks of the map $\mathbf{coim}\,(\varphi) \to \mathbf{im}\,(\varphi)$ are isomorphisms, i.e. $\mathbf{coim}\,(\varphi)_x \xrightarrow{\sim} \mathbf{im}\,(\varphi)_x$ for every $x \in X$ where the arrow is the stalk of the arrow on the bottom row of (3.2.4). By Theorem 3.1.3 we conclude that this arrow is an isomorphism:

$$\mathbf{coim}\,(\varphi) \xrightarrow[\text{via (3.2.4)}]{\sim} \mathbf{im}\,(\varphi). \tag{3.2.5}$$

We have the following theorem

Theorem 3.2.6. *$\mathcal{Sh}_X$ is an abelian category.*

Proof. It is clear that $\mathcal{Sh}_X$ is an additive category. The only non-trivial matter to be settled is that the natural maps from coimages to images are isomorphism, and that we have established. ☐

REFERENCES

[SP] The Stacks project authors, *The Stacks project*, `https://stacks.math.columbia.edu`, 2021.