LECTURE 8

Date of Lecture: Oct 19, 2021

The symbol $\mathscr{Ab}$ will denote the category of abelian groups. If X is a topological space, $\mathscr{Psh}_X$ and $\mathscr{Sh}_X$ denote the category of presheaves and the category of sheaves respectively on X. By a ring we mean a commutative ring with identity. The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. **Derived functors**

Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories.

1.1. **Enough injectives and enough projectives.**

We say $\mathscr{A}$ has *enough injectives* if every object in $\mathscr{A}$ can be embedded into an injective object of $\mathscr{A}$, i.e. given $A \in \mathscr{A}$, there exists a monomorphism $A \hookrightarrow E$ with E an injective object of $\mathscr{A}$. We say $\mathscr{A}$ has *enough projectives* if every object in $\mathscr{A}$ is the target of some epimorphism from a projective object, i.e. given $A \in \mathscr{A}$, there exists an epimorphism $P \twoheadrightarrow A$ with P projective.

1.1.1. If R is a ring then Mod_R has enough injectives as well as enough projectives, as we shall see below. Injectives in this category are injective modules, and projectives are projective modules, which are often characterised as direct summands of free modules.

If in the above case $R = \mathbf{Z}$, so that $\mathrm{Mod}_R = \mathscr{Ab}$, then $\mathbf{Q}/\mathbf{Z}$ is an injective object. Indeed, by *Baer's criterion,* it is enough to show that any group homomorphism $\phi\colon I \to \mathbf{Q}/\mathbf{Z}$, with an ideal I of $\mathbf{Z}$, can be extended to group homomorphism $\psi\colon \mathbf{Z} \to \mathbf{Q}/\mathbf{Z}$. If $I = 0$ this is obvious. So assume $I = \langle n \rangle \neq 0$ and suppose $\phi(n) = x + \mathbf{Z}$, $x \in \mathbf{Q}$. Define $\psi\colon \mathbf{Z} \to \mathbf{Q}/\mathbf{Z}$ by the formula $\psi(z) = (zx)/n + \mathbf{Z}$. Then ψ extends ϕ.

More generally, the same proof *mutatis mutandis* shows that if R is a PID and E is a *divisible* R-module, i.e., given $x \in E$ and $0 \neq r \in R$, there exists $y \in E$ such that $x = ry$, then E is an injective R-module.

Now suppose M is an abelian group. Pick generators m_λ, $\lambda \in \Lambda$ in M. We have a surjective map. $\oplus_{\lambda\in\Lambda}\mathbf{Z} \twoheadrightarrow M$ whence $M \cong \oplus_{\lambda\in\Lambda}\mathbf{Z}/K$ for some subgroup K of $\oplus_{\lambda\in\Lambda}\mathbf{Z}$. It follows that we have an inclusion $M \hookrightarrow \oplus_{\lambda\in\Lambda}\mathbf{Q}/K$. Since $\oplus_{\lambda\in\Lambda}\mathbf{Q}/K$ is divisible (same argument as that used for $\mathbf{Q}/\mathbf{Z}$), we see that $\mathscr{Ab}$ has enough injectives.

More generally, suppose R is a ring and M an R-module. Let $M_{\mathbf{Z}}$ be the underlying abelian group of M and E_0 an injective $\mathbf{Z}$-module containing $M_{\mathbf{Z}}$. Let $E := \mathrm{Hom}_{\mathbf{Z}}(R,\, M_{\mathbf{Z}})$. Then E is an R-module, with scalar multiplication being $r\phi = (s \mapsto \phi(rs))_{s\in R}$, $r \in R$, $\phi \in E$. The Hom-$\otimes$ adjointness formula yields:

$$\begin{aligned}\mathrm{Hom}_R(-,\, E) = \mathrm{Hom}_R(-,\, \mathrm{Hom}_{\mathbf{Z}}(R,\, E_{\mathbf{Z}})) &= \mathrm{Hom}_{\mathbf{Z}}(- \otimes_R R, E_{\mathbf{Z}})\\ &= \mathrm{Hom}_{\mathbf{Z}}(-,\, E_{\mathbf{Z}}).\end{aligned}$$

Since $E_{\mathbf{Z}}$ is an injective $\mathbf{Z}$-module, $\mathrm{Hom}_{\mathbf{Z}}(-,\, E_{\mathbf{Z}})$ is exact. Hence $\mathrm{Hom}_R(-,\, E)$ is exact. Thus E is injective. The natural map $M \to E$ given by $m \mapsto (r \mapsto rm)$ is injective. Hence Mod_R has enough injectives.

Mod_R also has enough projectives. In fact free modules are projective, since $\mathrm{Hom}_R(F,\, -)$ is exact for every free module F of Mod_R. Now given $M \in \mathrm{Mod}_R$ we can always find a free module F mapping onto M, which means Mod_R has enough projectives.

We will see later that $\mathcal{S}\!h_X$ and many of its most important subcategories have enough injectives.

Lemma 1.1.2. *Suppose $\mathcal{A}$ has enough injectives and X is an object of $\mathcal{A}$. Then we have X has a classical resolution $X \to E^\bullet$ with $E^\bullet$ an injective complex.*[1]

Proof. Let E^0 be any injective object which has A as a sub-object. We then have an exact sequence $0 \to A \to E^0$. Suppose we have selected $E^0, \dots, E^n$ so that they fit into an exact sequence

$$0 \longrightarrow A \longrightarrow E^0 \longrightarrow \dots \longrightarrow E^n.$$

Let B^{n+1} be the cokernel of $E^{n-1} \to E^n$. Pick E^{n+1} to be an injective object which has B^{n+1} as a sub-object and let $E^n \to E^{n+1}$ be the composite $E^n \twoheadrightarrow B^{n+1} \hookrightarrow E^{n+1}$. Then clearly we have extended the above exact sequence to

$$0 \longrightarrow A \longrightarrow E^0 \longrightarrow \dots \longrightarrow E^n \longrightarrow E^{n+1}.$$

By induction we are done. ☐

1.2. **Additive, left exact, right exact, and exact functors.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor. It is called *additive* if the natural map $\mathrm{Hom}_{\mathcal{A}}(X,\, Y) \to \mathrm{Hom}_{\mathcal{B}}(F(X),\, F(Y))$ is a homomorphism of abelian groups.[2] It is easy to see that if F is additive, then $F(X \oplus Y)$ is canonically isomorphic to $F(X) \oplus F(Y)$.[3] F is said to be *left exact* if it is additive and

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \quad \text{exact} \quad \Longrightarrow \quad 0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C) \quad \text{exact}.$$

It is *right exact* if it is additive and

$$A \longrightarrow B \longrightarrow C \longrightarrow 0 \quad \text{exact} \quad \Longrightarrow \quad F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow 0 \quad \text{exact}.$$

Finally F is *exact* if it is simultaneously right and left exact.[4] All these notions make sense even if F is contravariant by considering the associated covariant functor from $\mathcal{A}^\circ$ to $\mathcal{B}$.

1.2.1. For $X \in \mathcal{A}$, the functors $\mathrm{Hom}_{\mathcal{A}}(X,\, -)$ and $\mathrm{Hom}_{\mathcal{A}}(-,\, X)$ are left exact. If A is a ring, and $X \in \mathrm{Mod}_A$, then $(-) \otimes_A X\colon \mathrm{Mod}_A \to \mathrm{Mod}_A$ is right exact.

1.2.2. Suppose $F\colon \mathcal{A} \to \mathcal{B}$ is an additive functor, and $X^\bullet$ and $Y^\bullet$ complexes in $\mathcal{A}$, and $\phi\colon C^\bullet \to D^\bullet$ a map of complexes which is homotopic to zero. Then it is straightforward, from the definition of an additive functor, to see that the $F(\phi)\colon F(X^\bullet) \to F(Y^\bullet)$ is also homotopic to zero,

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[1] Injective complex = complex of injective objects.

[2] If F is contravariant, then we require that the corresponding covariant functor from the category $\mathcal{A}^\circ$ opposite to $\mathcal{A}$ to the category $\mathcal{B}$ be additive.

[3] The converse is also true. See Section 12.7 of [SP].

[4] One doesn't need the condition that F be additive in any of these definitions. It turns out that if F satisfies the second condition for left exactness (resp. right exactness) then F must be additive. See Section12.7 of [SP].

1.3. **An exactness lemma.** Let $F\colon \mathscr{A} \to \mathscr{B}$ be an additive functor. Let $E^\bullet$ be a bounded below exact injective complex. Then the identitiy map $\mathbf{1}_{E^\bullet}\colon E^\bullet \to E^\bullet$ is homotopic to zero by Proposition 2.1.2 of of Lecture 6. From the remark in 1.2.2, it follows that $\mathbf{1}_{F(E^\bullet)} = F(\mathbf{1}_{E^\bullet})$ is homotopic to zero. Thus $\mathrm{H}^p(F(E^\bullet)) = 0$ for $p \in \mathbf{Z}$ since the identity map on each $\mathrm{H}^p(F(E^\bullet))$ is zero. We have thus proved:

Lemma 1.3.1. *If $E^\bullet$ is a bounded below exact injective complex in $\mathscr{A}$ and $F\colon \mathscr{A} \to \mathscr{B}$ an additive functor, then $F(E^\bullet)$ is exact.*

1.4. **Right derived functors of left exact functors.** Let $F\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor. Suppose $\mathscr{A}$ has enough injectives. Let $A \in \mathscr{A}$. By Lemma 1.1.2 we know that A has an injective resolution by a bounded below complex (in fact a classical injective resolution). If $A \to E^\bullet$ and $A \to I^\bullet$ are two bounded below injective resolutions of A, then by Proposition 2.3.5 of Lecture 7 we have an isomorphism $\varphi\colon E^\bullet \xrightarrow{\sim} I^\bullet$ in $\mathbf{K}(\mathscr{A})$ such that the the following diagram commutes in $\mathbf{K}(\mathscr{A})$

(1.4.1)
$$\begin{tikzcd}
 & E^\bullet \arrow[dd, "\varphi", "\wr"'] \\
A \arrow[ur] \arrow[dr] & \\
 & I^\bullet
\end{tikzcd}$$

Since additive functors preserve homotopies, $F(\varphi)\colon F(E^\bullet) \to F(I^\bullet)$ is well defined in $\mathbf{K}(\mathscr{B})$. Let $\psi\colon I^\bullet \to E^\bullet$ be the inverse of φ (in $\mathbf{K}(\mathscr{A})$). Then clearly, $F(\psi)$ is the inverse of $F(\varphi)$ in $\mathbf{K}(\mathscr{B})$. In particular $F(\varphi)$ is an isomorphism in $\mathbf{K}(\mathscr{B})$ whence we have

$$\mathrm{H}^p(F(\varphi))\colon \mathrm{H}^p(F(E^\bullet)) \xrightarrow{\sim} \mathrm{H}^p(F(I^\bullet)), \qquad p \in \mathbf{Z}. \tag{1.4.2}$$

Moreover if $A \to J^\bullet$ is a third injective resolution with $J^\bullet$ bounded below, and we denote, for good book-keeping, the isomorphism φ above by φ_{IE}, then, in an obvious notation,

$$\varphi_{IE} \circ \varphi_{EJ} = \varphi_{IJ}. \tag{1.4.3}$$

Note that what we called ψ earlier is φ_{EI} in this notation.

There is another way of showing the isomorphism in (1.4.2). The technique is interesting enough to be given exposure now, since it is used often. Pick a representative of φ in $\mathbf{C}(\mathscr{A})$, and for simplicity, denote it too by φ. The mapping cone $C_\varphi^\bullet$ is a bounded below exact complex of injectives. It is exact because φ is a quasi-isomorphism. It is a complex of injectives because (finite) direct sums of injectives are injective. Now, from the definition of a mapping cone, clearly $F(C_\varphi^\bullet)$ is the mapping cone of $F(\varphi)$, i.e. $F(C_\varphi^\bullet) = C_{F(\varphi)}^\bullet$. By Lemma 1.3.1, $F(C_\varphi^\bullet)$ is exact. Hence $C_{F(\varphi)}^\bullet$ is exact, whence $F(\varphi)$ is a quasi-isomorphism, giving (1.4.2).

Definition 1.4.4. Let $A \in \mathscr{A}$ and pick a bounded below injective resolution $A \to E^\bullet$. Let $p \in \mathbf{Z}$. The p^{th} *right derived functor* $\mathrm{R}^pF(A)$ of F at A is the object in $\mathscr{B}$ defined by $\mathrm{R}^pF(A) = \mathrm{H}^p(F(E^\bullet))$.

In view of the discussion above, $\mathrm{R}^pF(A)$ is well defined up to canonical isomorphism. Moreover, since classical injective resolutions of A exist, using one to compute $\mathrm{R}^pF(A)$, we see that

$$\mathrm{R}^pF(A) = 0, \qquad p < 0. \tag{1.4.5}$$

We have used the term functor in Definition 1.4.4. Let us show that the R^pF is a functor as A varies. Let $f\colon A \to A'$ be a map in $\mathscr{A}$ and let $\alpha\colon A \to E^\bullet$ and $\beta\colon A' \to I^\bullet$ be bounded below injective resolutions. We then have the composite $g\colon A \to I^\bullet$ given by $g = \beta\circ f$. Apply Proposition 2.1.2 of Lecture 7 to get a unique map $h\colon E^\bullet \to I^\bullet$ in $\mathbf{K}(\mathscr{A})$ such that $h\circ\alpha = g = \beta\circ f$. Once again using the fact that additive functors preserve homotopies, we get a well defined map $F(h)\colon F(E^\bullet) \to F(I^\bullet)$ in $\mathbf{K}(\mathscr{B})$. Taking the p^{th} cohomology of $F(h)$ we get:

$$\mathrm{R}^pF(f)\colon \mathrm{R}^pF(A) \longrightarrow \mathrm{R}^pF(A'), \qquad p \in \mathbf{Z}. \tag{1.4.6}$$

We leave it to the reader to check that this gives an additive functor, one for each $p \in \mathscr{A}$

$$\mathrm{R}^pF\colon \mathscr{A} \to \mathscr{B} \tag{1.4.7}$$

1.4.8. To make everything canonical, we should once and for all choose bounded below injective resolutions $A \to E^\bullet_A$, one for each A (this involves a humongous axiom of choice), and define $\mathrm{R}^pF(A)$ using $E^\bullet_A$. This is usually implicit when we work with derived functors. And locally, if we find in the course of an argument that we should pick a convenient bounded below injective resolution (i.e. convenient for that argument), then the implicit understanding is that we are using the canonical isomorphism between $E^\bullet_A$ and the resolution picked for local convenience.

1.5. **The δ-functor property.** Let $F\colon \mathscr{A} \to \mathscr{B}$ be left exact and suppose we have a short exact sequence

$$0 \longrightarrow A' \xrightarrow{\ u\ } A \xrightarrow{\ v\ } A'' \longrightarrow 0 \tag{1.5.1}$$

in $\mathscr{A}$. Let $A' \to E^\bullet$ and $A \to I^\bullet$ be bounded below injective resolutions and $\varphi\colon E^\bullet \to I^\bullet$ the unique map in $\mathbf{K}(\mathscr{A})$ which lifts u. For convenience, though this is not necessary, let the two injective resolutions be *classical* resolutions, so that $E^p = I^p = 0$ for $p < 0$. Pick a representative of φ in $\mathrm{Hom}_{\mathbf{C}(\mathscr{A})}(E^\bullet, I^\bullet)$ and for simplicity continue to call it φ. Then $C^\bullet_\varphi$ is a complex of bounded below injectives. The short exact sequence

$$0 \longrightarrow I^\bullet \longrightarrow C^\bullet_\varphi \longrightarrow E^\bullet[1] \longrightarrow 0 \tag{1.5.2}$$

associated with the mapping cone $C^\bullet_\varphi$ is split at each degree and hence we have a short exact sequence of complexes

$$0 \longrightarrow F(I^\bullet) \longrightarrow F(C^\bullet_\varphi) \longrightarrow F(E^\bullet)[1] \longrightarrow 0. \tag{1.5.3}$$

The long exact sequence associated with (1.5.2) gives us that $\mathrm{H}^p(C^\bullet_\varphi) = 0$ for $p \geq 1$, since it is caught between $\mathrm{H}^p(I^\bullet) = 0$ and $\mathrm{H}^p(E^\bullet[1]) = \mathrm{H}^{p+1}(E^\bullet) = 0$. The same argument shows that for $p \leq -2$, $\mathrm{H}^p(C^\bullet_\varphi) = 0$. The objects $\mathrm{H}^{-1}(C^\bullet_\varphi)$ and $\mathrm{H}^0(C^\bullet_\varphi)$ fit into the exact sequence

$$0 \longrightarrow \mathrm{H}^{-1}(C^\bullet_\varphi) \longrightarrow \mathrm{H}^{-1}(E^\bullet[1]) \longrightarrow \mathrm{H}^0(I^\bullet) \longrightarrow \mathrm{H}^0(C^\bullet_\varphi) \longrightarrow \mathrm{H}^0(E^\bullet[1])$$

Now $\mathrm{H}^{-1}(E^\bullet[1]) = \mathrm{H}^0(E^\bullet) = A'$, $\mathrm{H}^0(I^\bullet) = A$, and $\mathrm{H}^0(E^\bullet[1]) = \mathrm{H}^1(E^\bullet) = 0$, the above exact sequence gives us via (1.3.2) of Lecture 7 an exact sequence

$$0 \longrightarrow \mathrm{H}^{-1}(C^\bullet_\varphi) \longrightarrow A' \xrightarrow{\ u\ } A \longrightarrow \mathrm{H}^0(C^\bullet_\varphi) \longrightarrow 0.$$

Now $u\colon A' \to A$ is injective and hence $\mathrm{H}^{-1}(C_\varphi^\bullet) = 0$. We thus have an exact sequence

$$0 \longrightarrow A' \xrightarrow{u} A \longrightarrow \mathrm{H}^0(C_\varphi^\bullet) \longrightarrow 0,$$

which means via (1.5.1) that $\mathrm{H}^0(C_\varphi^\bullet) \xrightarrow{\sim} A''$. We have thus proved that $\mathrm{H}^p(C_\varphi^\bullet) = 0$ for $p \neq 0$ and $\mathrm{H}^0(C_\varphi^\bullet) \xrightarrow{\sim} A''$. Since $C_\varphi^\bullet$ is a bounded below injective complex, this actually shows that $C_\varphi^\bullet$ is an injective resolution of A''. Here are the details of that well known fact (under the simplifying assumption that $E^\bullet$ and $I^\bullet$ are classical resolutions). Since $E^\bullet$ and $I^\bullet$ are classical resolutions, therefore $C_\varphi^p = 0$ for $p \leq -2$ and hence $C_\varphi^\bullet$ can be displayed as

$$0 \longrightarrow C_\varphi^{-1} \longrightarrow C_\varphi^0 \longrightarrow C_\varphi^1 \longrightarrow \dots \longrightarrow C_\varphi^n \longrightarrow \dots$$

Since $\mathrm{H}^{-1}(C_\varphi^\bullet) = 0$, this means C_φ^{-1} is a sub-object of C_φ^0. Now use the fact that C_φ^{-1} is injective to conclude that it is a direct summand of C_φ^0.[5] Thus $C_\varphi^0 = C_\varphi^{-1} \oplus J$ where J is an injective object being the direct summand of an injective object, namely C_φ^0.[6] Let us write Z^p for $Z^p(C_\varphi^\bullet)$ and $B^p = B^p(C_\varphi^\bullet)$. It is then easy to see that $B^0 = C_\varphi^{-1}$ and since injective subobjects split off as direct summands, we have the decomposition of $Z^0 = B^0 \oplus H$ and H is isomorphic to $Z^0/B^0 = \mathrm{H}^0(C_\varphi^\bullet)$ hence A'' (which is isomorphic to $\mathrm{H}^0(C_\varphi^\bullet)$) can be regarded as a sub-object of Z^0. The map $A'' \hookrightarrow Z^0 \subset C_\varphi^0$ gives us a map of complexes $A'' \to C_\varphi^\bullet$ which is a quasi-isomorphism. In other words $C_\varphi^\bullet$ is an injective resolution of A''.

Now consider the long exact sequence associated with (1.5.3). This gives us via (1.3.2) of Lecture 7 a long exact sequence

$$\text{(1.5.4)} \qquad \dots \xrightarrow{\delta^{n-1}} \mathrm{R}^n F(A') \xrightarrow{u} \mathrm{R}^n F(A) \xrightarrow{v} \mathrm{R}^n F(A'') \xrightarrow{\delta^n} \mathrm{R}^{n+1} F(A') \xrightarrow{u} \dots$$

where the labels u and v are shorthand for "via u" and "via v", and the maps δ^p the connecting homomorphisms.

1.5.5. There was no need to assume $A' \to E^\bullet$ and $A \to I^\bullet$ were classical injective resolutions. Even if they were simply bounded below injective complexes, we would still get $\mathrm{H}^p(C_\varphi^\bullet) = 0$ for $p \neq 0$ and $\mathrm{H}^0(C_\varphi^\bullet) \xrightarrow{\sim} A''$. Using essentially the technique we used, one can show (by starting at the lowest degree $C_\varphi^\bullet$ is nonzero) that $C_\varphi^0 = B^0 \oplus J$, $Z^0 = B^0 \oplus (Z^0/B^0) = B^0 \oplus \mathrm{H}^0(C_\varphi^\bullet)$, whence A'' can be identified with a submodule of Z^0, and the resulting composite $A'' \hookrightarrow Z^0 \hookrightarrow C_\varphi^0$ gives a map of complexes $A'' \to C_\varphi^\bullet$ which is a quasi-isomorphism. If you are interested, fill in the details. Injectivity of $C_\varphi^\bullet$ plays a major role in these proofs.

We have thus proved the following theorem:

Theorem 1.5.6. *Let $\mathscr{A}$ have enough injectives, and let $F\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor. Then a short exact sequence of the form* (1.5.1) *in $\mathscr{A}$ induces the long exact sequence* (1.5.4) *in $\mathscr{B}$.*

1.5.7. The long exact sequence (1.5.4) is proved classically using the *Horseshoe Lemma*. However, I feel it is important to quickly remove the crutch of classical

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[5] The splitting map $C_\varphi^0 \to C_\varphi^{-1}$ being any inverse image of the identity $\mathbf{1}_{C_\varphi^{-1}}$ under the surjective map $\mathrm{Hom}_{\mathscr{A}}(C_\varphi^0, C_\varphi^{-1}) \twoheadrightarrow \mathrm{Hom}_{\mathscr{A}}(C_\varphi^{-1}, C_\varphi^{-1})$.

[6] One has $\mathrm{Hom}_{\mathscr{A}}(-, C_\varphi^0) = \mathrm{Hom}_{\mathscr{A}}(-, C_\varphi^{-1}) \oplus \mathrm{Hom}_{\mathscr{A}}(-, J)$, and since the functor on the left is exact, this forces the exactness of the two summands on the right.

resolutions, and hence my emphasis on mapping cones and arguments involving that.

References

[SP] The Stacks project authors, *The Stacks project*, https://stacks.math.columbia.edu, 2021.