LECTURE 6

Date of Lecture: Oct 7, 2021

The symbol $\mathcal{Ab}$ will denote the category of abelian groups. If X is a topological space, $\mathcal{Psh}_X$ and $\mathcal{Sh}_X$ denote the category of presheaves and the category of sheaves respectively on X. By a ring we mean a commutative ring with identity. The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Intuitive ideas about $\mathbb{P}^n_k$

There are no proofs here. This section is more to shore up your intuition about projective spaces and their subschemes. We fix a field k, and for simplicity assume it is algebraically closed. We use the equivalence of categories between quasi-projective varieties over k in the classical sense and certain schemes associated with the Proj of a graded ring, as done in Lecture 5 and without comment move from one category to the other. The point is to shore up intuition, without getting caught up in language. Note that by the nullstellensatz, if $\mathbb{P}^n_k$ is thought of as a scheme, then the classical version of projective n-space is the set of closed points of the scheme $\mathbb{P}^n_k = \operatorname{Proj}(k[X_0, \ldots, X_n])$ under the correspondence given in Lecture 5.

1.1. **Projective space as the space of lines.** The space $\mathbb{P}^n_k$ is the space of lines through the origin in k^{n+1}. Let $X_0, \ldots, X_n$ denote the coordinate functions of k^{n+1} which we identify with $\mathbb{A}^{n+1}_k$. Given a line through the origin in k^{n+1}, so long as it does not lie in the hyperplane $X_0 = 0$, it meets a unique point in the affine hyperplane $X_0 = 1$. We can identify the hyperplane $X_0 = 1$ with $\mathbb{A}^n_k$, and therefore the points of $\mathbb{P}^n_k$ which are not lines in $X_0 = 0$ can be regarded as affine n-space, and call this subset U_0. Similarly U_i can be defined as those lines through the origin in k^{n+1} which do not lie on the hyperplane $X_i = 0$, and since these meet the hyperplane $X_i = 1$ in exactly one point, U_i can also be identified with $\mathbb{A}^n_k$. Thus we can cover $\mathbb{P}^n_k$ by $n+1$ sets each of which can be identified with $\mathbb{A}^n_k$. One checks easily that the "transition functions" on U_i to U_j are algebraic functions (they are of the form X_j/X_i).

1.2. **Cones.** An algebraic set C in $\mathbb{A}^{n+1}_k$ is called a *cone* if it has the following property – if a point P which is not the origin lies on V then the entire line through the origin passing through P lies on C. Cones are the zero sets of systems of homogeneous polynomial equations in $n+1$ variables $X_0, \ldots, X_n$. That such a system of equations gives a cone is seen by the fact that if f is homogeneous of degree d and f vanishes at $P = (\alpha_0, \ldots, \alpha_n)$ then $f(t\alpha_0, \ldots, t\alpha_n) = t^d f(\alpha_0, \ldots, \alpha_n) = 0$ for all $t \in k$. In particular, if P is not the origin, then the line ℓ containing P and the origin lies in C. What is interesting is that if an algebraic set C is a cone, then one can always find homogenous polynomials $f_1, \ldots, f_r$ such that $C = Z(f_1, \ldots, f_r)$. Indeed, suppose $f \in k[X_0, \ldots, X_n]$ vanishes on C. Let $f = p_0 + p_1 + \cdots + p_d$ where p_i

is a homogeneous polynomial of degree i for $i = 1, \ldots, d$. Let $\boldsymbol{x} = (x_o, \ldots, x_n) \in C$. Then $t\boldsymbol{x}$ for all $t \in k$. Which means $f(t\boldsymbol{x}) = 0$ for every $t \in k$. This means

$$p(\boldsymbol{x}) + tp_1(\boldsymbol{x}) + t^2 p_2(\boldsymbol{x}) + \cdots + t^d p_d(\boldsymbol{x}) = 0$$

for all $t \in k$. Since k is algebraically closed it is an infinite field. The above relation then shows that $p_i(\boldsymbol{x}) = 0$ for all $i = 1, \ldots, d$. Since the p_i are homogenous and vanish on all points of C, it follows that C is given by the vanishing of homogeneous polynomials.

The origin of $\mathbb{A}_k^{n+1}$ must belong to the cone C. This is called the *vertex* of the cone C. Assume C is not a singleton. Then the intersection of C with the hyperplane P_0 given by $X_0 = 1$ can be identified with an algebraic set V in $\mathbb{A}_k^n$. Indeed, the locus $X_0 = 1$ can be identified with $\mathbb{A}_k^n$, and if f is a polynomial vanishing on C, then the polynomial $g(X_1, \ldots, X_n) := f(1, X_1, \ldots, X_n)$ vanishes on $V_0 = P_0 \cap C$. and conversely if $g(X_1, \ldots, X_n)$ is a polynomial vanishing on $V_0 = P_0 \cap C$, then the polynomial $f(X_0, \ldots, X_n) := X_0^{\deg g} g(X_1/X_0, \ldots, X_n/X_0)$ is homogenous of degree equal to $\deg g$ and vanishes on C. Once again, one can extend this argument to intersections of C with the hyperplanes P_i given by $X_i = 1$, to arrive at algebraic sets $V_i = C \cap P_i$ in $\mathbb{A}_k^n$. The algebraic sets V_i "glue" together to form an algebraic set V in $\mathbb{P}_k^n$.

1.3. **Points at "infinity".** Let us keep the notations of the previous subsection.

Note that for $i = 0, \ldots, n$, the hyperplanes Π_i, given by the equation $X_i = 0$ are also cones, and by the above discussion we have algebraic sets H_i in $\mathbb{P}_k^n$ corresponding to Π_i. In fact a little thought shows that each H_i can be identified with $\mathbb{P}_k^{n-1}$.

More generally, if $\Pi \subset \mathbb{A}_k^{n+1}$ is the cone given by an equation of the form $\sum_{i=0}^n a_i X_i = 0$ with $a_i \in k$, and at least one a_i non-zero, and H is the projective hyperplane in $\mathbb{P}_k^n$ corresponding to Π, then the complement $U = \mathbb{P}_k^n \smallsetminus H$ can be identified with $\mathbb{A}_k^n$ and H with $\mathbb{P}^n$. Indeed, consider the algebraic set P in $\mathbb{A}_k^{n+1}$ given by $\sum_{i=0}^n a_i X_i = d$ with $a_i \in k$ and d some non-zero element in k (the favourite choice is $d = 1$). Note that P is a non-trivial translate of Π. Now U is the set of lines through the origin in $\mathbb{A}_k^{n+1}$ which do not lie in Π, and by identifying such lines with their intersection with P, we can identify U with P, and P can clearly be identified with $\mathbb{A}_k^n$.

From the above discussion, we can break up $\mathbb{P}_k^n$ as the disjoint union $\mathbb{P}_k^n = U \coprod H$, and hence in a sense as $\mathbb{P}_k^n = \mathbb{A}_k^n \coprod \mathbb{P}_k^{n-1}$.

If we fix our model of $\mathbb{A}_k^n$ as U, then the points on $H = \mathbb{P}_k^{n-1}$ are called *points at infinity* and one calls H as the *hyperplane at infnity*. Note this depends on the choice of Π. Indeed any linear hyperplane in $\mathbb{P}_k^n$ can be used to play the role of the hyperplane at infinity and its complement as the affine space $\mathbb{A}_k^n$ of interest.

Let us fix U and H as above, and when convenient make the identifications $U = \mathbb{A}_k^n$ and $H = \mathbb{P}_k^{n-1}$. If V_{aff} is an affine algebraic set in $U = \mathbb{A}_k^n$, then by the procedure given above for getting a homogeneous polynomial from a non-homogeneous one (modified suitably), we can find a projective algebraic set V such that $V \cap U = V_{\text{aff}}$. The points of $V \smallsetminus V_{\text{aff}}$ are then called *the points at infinity of* V. We once again point out that the idea of points at infinity depends upon on our choice of H.

1.3.1. The Example of an affine hyperboloid. The above ideas are illustrated by the following example. Consider the hyperbola

$$X^2 - Y^2 = 1 \tag{1.3.1.1}$$

in the XY-plane. Homogenising it in three variables we get

$$X^2 - Y^2 - Z^2 = 0. \tag{1.3.1.2}$$

This is a cone. The relationship between this cone and the original hyperbola is as in the pictures that follow.

In Figure 1 the arrow pointing out of the page is the Y-axis, the upward arrow the Z-axis. The brown plane is the one given by $Z = 1$ and the grey one the XY-plane, i.e. the plane $Z = 0$. The cone is (1.3.1.2). The red curve can be identified with the affine hyperbola in (1.3.1.1). It is the intersection of the displayed cone with the plane $Z = 1$. The lines through the origin in the $Z = 0$ plane form the projective line "at infinity". On $Z = 0$ one can see two purple lines. These correspond to two points "at infinity" on the projective variety "completing" the affine hyperbola. Note that the two purple lines are asymptotes to the affine hyperbola, and they "meet" the projectivisation "at infinity" in the two points represented by the these lines.

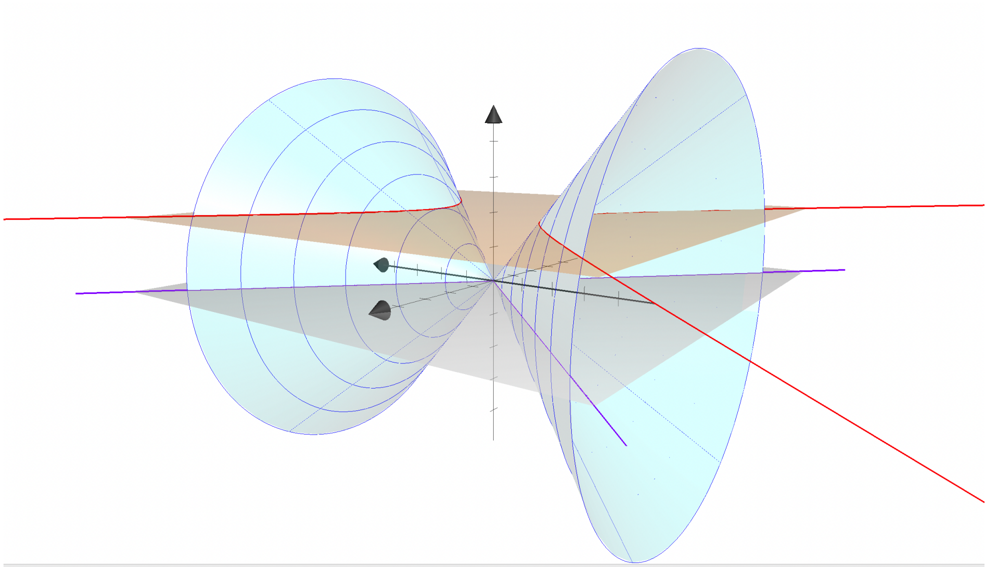


FIGURE 1.

Figure2 displays the affine hyperbola and its two points at infinity (the two purple lines through the origin), without the cone (1.3.1.2). Taken together, they form the projective completion of the affine hyperbola. The two points at infinity, when interpreted as lines in the XY-plane, are the asymptotes of (1.3.1.1)

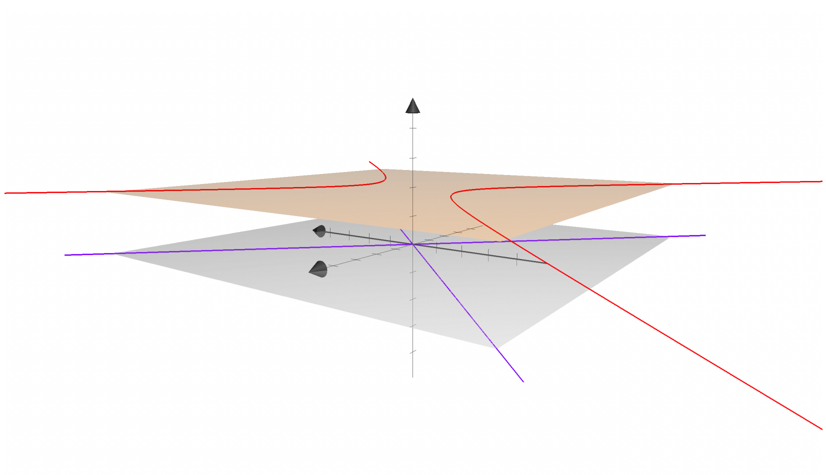


FIGURE 2. Projective hyperbola = affine hyperbola + two points at infinity.

Things can look different in another "coordinate chart". If we take our hyperplane at infinity to be the plane $X = 0$ and use $X = 5/2$ as the model for $\mathbb{A}^2_{\mathbf{R}}$, then situation looks as in Figure 3. The green plane is the model for $\mathbb{A}^2_{\mathbf{R}}$ and now the affine portion of (1.3.1.2) is green circle on the picture. There are no real points at infinity though there two complex points at infinity.

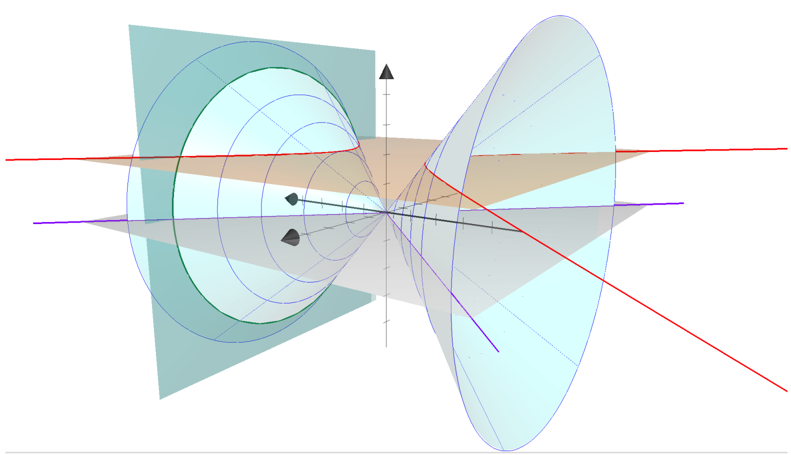


FIGURE 3. The green plane is $X = 5/2$ and the green circle is how our variety looks in this chart. It has no real points at infnity, but it does have two complex points at infinity.

In Figure 4 we have both co-ordinate systems, and the map f from the affine hyperbola to the circle is our "transition function". The idea is to take a point P on the red hyperbola draw a from that point to the origin, extend the line until it intersects the green circle. The point on the circle is $f(P)$.

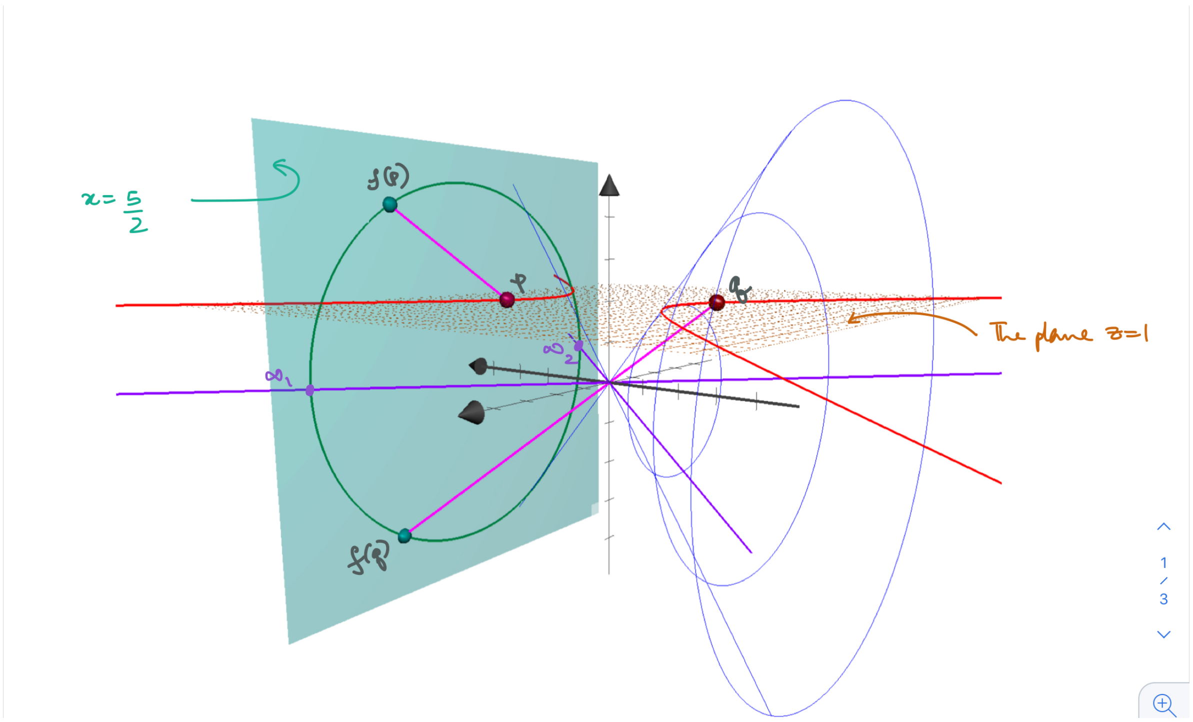


FIGURE 4. The points p and q on the red hyperbola correspond to $f(p)$ and $f(q)$ on the green circle. The circle has two points which are not on the red hyperbola, namely its intersection with $Z = 0$. These points are labelled in the picture as ∞_1 and ∞_2. These points correspond to the two points at infinity of the projectivisation of the hyperbola (1.3.1.1).

Finally Figure 5 is a clean picture of the above situation, without the cone interfering.

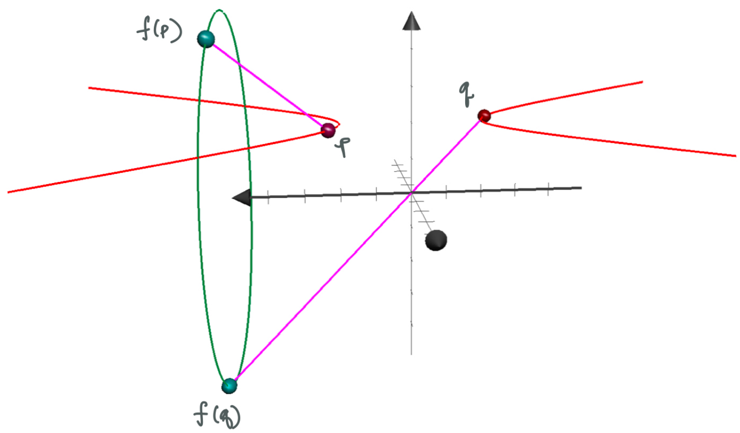


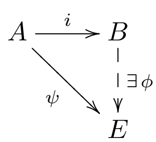
FIGURE 5.

2. **Injective objects and complexes**

Fix an abelian category $\mathcal{A}$ for this section.

2.1. **Injectives.** Let E be an object in $\mathcal{A}$. The following conditions on E are equivalent (easy to verify – use elements if you feel like):

1. $\mathrm{Hom}_{\mathcal{A}}(-, E)$ is exact on $\mathcal{A}$.
2. For any injective map $i\colon A \to B$ in $\mathcal{A}$, the induced map $\mathrm{Hom}_{\mathcal{A}}(B, E) \to \mathrm{Hom}_{\mathcal{A}}(A, E)$ given by $\phi \mapsto \phi \circ i$ is surjective.
3. Given an injective map $i\colon A \to B$ in $\mathcal{A}$ and a map $\psi\colon A \to E$ in $\mathcal{A}$, there is a map $\phi\colon B \to E$ such that $\phi \circ i = \psi$. In other words there exists a ϕ which fills the dashed arrow below (to make the diagram commute).



E is said to be an *injective object* of $\mathcal{A}$ if it satisfies any of the equivalent conditions above.

2.1.1. A projective object of $\mathcal{A}$ is an injective object of the opposite category $\mathcal{A}^\circ$. Work out a definition which does not mention the opposite category $\mathcal{A}^\circ$.

The following Proposition is the key to many things in derived categories. Recall that a map of complexes $\phi\colon C^\bullet \to D^\bullet$ is said to be homotopic to zero, written $\phi \sim 0$, if there exist maps $k^p\colon C^p \to D^{p-1}$, $p \in \mathbf{Z}$, such that $d_D^{p-1} \circ k^p + k^{p+1} \circ d_C^p = \phi$.

Proposition 2.1.2. *Let $C^\bullet$ be bounded below exact sequence in $\mathcal{A}$, $E^\bullet$ a bounded below complex of injectives in $\mathcal{A}$, and $\phi\colon C^\bullet \to E^\bullet$ a map of complexes. Then $\phi \sim 0$.*

Proof. There is an integer N such that $C^p = E^p = 0$ for all $p < N$. By shifting our complexes by $|N|$ units (to the left if $N \geq 0$ and to the right if $N < 0$) if necessary, we may assume $C^p = E^p = 0$ for $p < 0$. We have a commutative diagram of complexes with the top row exact:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & C^0 & \xrightarrow{d_C^0} & C^1 & \xrightarrow{d_C^1} & \cdots\cdots & \xrightarrow{d_C^{n-2}} & C^{n-1} & \xrightarrow{d_C^{n-1}} & C^n & \xrightarrow{d_C^n} & \cdots \\
 & & {\scriptstyle \phi^0}\downarrow & & {\scriptstyle \phi^1}\downarrow & & & & {\scriptstyle \phi^{n-1}}\downarrow & & {\scriptstyle \phi^n}\downarrow & & \\
0 & \longrightarrow & E^0 & \xrightarrow[d_E^0]{} & E^1 & \xrightarrow[d_E^1]{} & \cdots\cdots & \xrightarrow[d_E^{n-2}]{} & E^{n-1} & \xrightarrow[d_E^{n-1}]{} & E^n & \xrightarrow[d_E^n]{} & \cdots
\end{array}$$

Set $k^i = 0$ for i negative. Since the top row is exact, $C^0 \to C^1$ is an injective map. Since E^0 is an injective object, we get a map $k^1\colon C^1 \to E^0$ such that $k^1 \circ d_C^0 = \phi^0$. Let n be a positive integer. Suppose k^i have been defined for $i \leq n-1$ so that the homotopy condition is satisfied up to the $(n-2)^{\text{th}}$ stage. We have

$$d_E^{n-2} \circ (\phi^{n-2} - (k^{n-1} \circ d_C^{n-2})) = d_E^{n-2} \circ (d_E^{n-3} \circ k^{n-2}) = (d_E^{n-2} \circ d_E^{n-3}) \circ k^{n-2} = 0$$

whence

$$d_E^{n-2} k^{n-1} d_C^{n-2} = d_E^{n-2} \phi^{n-2} = \phi^{n-1} d_C^{n-2}.$$

The image of d_C^{n-2} is $B^{n-1}(C^\bullet) = Z^{n-1}(C^\bullet)$, the latter equality due to the fact that $C^\bullet$ is exact. Thus the map

$$C^{n-2} \xrightarrow{d_C^{n-2}} Z^{n-1}(C^\bullet)$$

is an epimorphism. Therefore, using the fact that $d_E^{n-2} k^{n-1} d_C^{n-2} = \phi^{n-1} d_C^{n-2}$, we get that $d_E^{n-2} k^{n-1}|_{Z^{n-1}(C^\bullet)} = \phi^{n-1}|_{Z^{n-1}(C^\bullet)}$. In other words

$$(\phi^{n-1} - d_E^{n-2} k^{n-1})|_{Z^{n-1}(C^\bullet)} = 0.$$

It follows that there is a map $\kappa\colon C^{n-1}/Z^{n-1}(C^\bullet) \to E^{n-1}$ such that the following diagram commutes:

$$\begin{array}{ccc}
C^{n-1} & \twoheadrightarrow & C^{n-1}/Z^{n-1}(C^\bullet) \\
{\scriptstyle \phi^{n-1} - d_E^{n-2} k^{n-1}}\downarrow & \swarrow{\scriptstyle \kappa} & \\
E^{n-1} & &
\end{array}$$

Now $C^{n-1}/Z^{n-1}(C^\bullet) = B^n(C^\bullet) = Z^n(C^\bullet)$ is a subobject of C^n. Since E^{n-1} is injective, therefore κ can be "extended" from $Z^n(C^\bullet)$ to C^n giving us a map $k^n\colon C^n \to E^{n-1}$. By the construction of k^n we have $k^n d_C^{n-1} + d_E^{n-2} k^{n-1} = \phi^{n-1}$. In other words we have defined k^n so that the homotopy condition for ϕ extends to the $(n-1)^{\text{th}}$ stage. This completes the proof. □