# Lecture 5:

Let us consider the polynomial ring $k[x_1, \ldots, x_n]$, where $k$ is an alg. closed field.

For a element $f \in k[x_1, \ldots, x_n]$ we define the zero set of $f$ as

$$Z(f) = \{ P \in \mathbb{A}^n \mid f(P) = 0 \}$$

For a subset $T \subseteq k[x_1, \ldots, x_n]$ we define the zero set of $T$ as

$$Z(T) = \{ P \in \mathbb{A}^n \mid f(P) = 0 \ \forall f \in T \}$$

A subset $Y \subseteq \mathbb{A}^n$ is called an algebraic set if

$$Y = Z(T), \quad T \subseteq k[x_1, \ldots, x_n]$$

We define the topology on $\mathbb{A}^n$ by taking as closed ~~sets~~ subsets the sets of the form $Z(T)$, $T \subseteq k[x_1, \ldots, x_n]$.

An affine algebraic variety is an irreducible closed subset in $\mathbb{A}^n$.

A regular map:

Let $Y$ be an affine variety over $k$. A map $f: Y \to k$ is said to be regular at $P$, if for each $P \in Y$. there is a open nbd $U$ of $P$ and elements $g, h \in k[x_1, \dots, x_n]$ s.t. $h$ is nowhere zero on $U$ & $f = \frac{g}{h}$ on $U$.

A variety over $k$ is an affine or quasi-affine or projective or quasi-projective varity.

~~An~~ A quasi-affine variety is an open subset of an affine variety.

Let $S$ be the graded ring $k[x_0, \dots, x_n]$.

Then, for a homogenous polynomial $f \in S$, we define the zero set of $f$ as.

$$Z(f) = \{ P \in \mathbb{P}^n \mid f(P) = 0 \}$$

For a subset $T$ of homogenous elements in $S$, we define the zero set of $T$ as

$\mathbb{P}^n \cong \mathbb{A}^{n+1} \setminus \{0\} / \sim$

$(a_0, \dots, a_n) \sim (\lambda a_0, \dots, \lambda a_n)$

$\lambda \neq 0$

Zero

$Z(T) = \{ P \in \mathbb{P}^n \mid f(P) = 0 \ \forall f \in T \}$.

III We define the topology on $\mathbb{P}^n$ by taking as closed subset the sets of the form $Z(T)$, $T$ is any set of homogenous elements in $S$.

IV A projective variety is an irr. closed subset of $\mathbb{P}^n$.

VI Let $Y$ be a projective variety over $k$. Then a map $f: Y \to R$ is regular if for each $P \in Y$, there is a open nbd $U$ of $P$ and homogenous elements $g, h \in S$ of same degree s.t. $h$ is nowhere 0 on $U$ and $f = \frac{g}{h}$ on $U$.

$f$ is said to be regular on $Y$ if it is regular at each point $P$ of $Y$.

VII Now, let $V$ be a variety over $k$. For an open subset $U \subseteq V$, let, $\mathcal{O}(U)$ be the set of regular functions on $U$.

... subset $W \subseteq U$ the map

For open subsets [...] $\rho_{UW} : \mathcal{O}(U) \longrightarrow \mathcal{O}(W)$ is the natural restriction map.

Then $\mathcal{O}$ is a sheaf of rings on $V$, and is called the sheaf of regular functions on $V$.

Let $S$ be a fixed scheme. <u>A scheme over $S$</u> is a scheme $X$, together with the morphism $X \to S$. For any two schemes $X$ and $Y$, a morphism $f : X \longrightarrow Y$ is said to be a $S$-morphism if it is compatible with the given morphisms to $S$. i.e., the following diagram commutes

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

We denote by $Sch_S$, the category of schemes over $S$.

<u>Theorem</u> :-

Let $k$ be an alg. closed field.

[...] fully

Let $k$ ... natural ...

There is a natural fully faithful function $t: Var(k) \to Sch_k$ from the category of varieties over $k$ to the category of schemes over $k$. For any variety $V$, its topological space is homeomorphic to the set of closed points of $sp(t(V))$ and its sheaf of regular function is obtained by restricing structure sheaf of $t(V)$ via this homeomorphism.

Proof :-

Let $X$ be any topological space & let $t(X)$ be the set of all irreducible closed subsets of $X$.

- If $Y$ is a closed subset of $X$ then $t(Y) \subseteq t(X)$.

- for two closed subsets $Y_1$ and $Y_2$ of $X$
$$t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2).$$

- For any family $\{Y_i\}$ of closed

subsets of $X$.

$$t(\cap Y_i) = \cap t(Y_i)$$

So, we define a topology on $t(X)$ by taking as closed sets, the subsets of the form $t(Y)$, where $Y$ is a closed subset of $X$.

~~Als~~ For, any cts map $f : X_1 \to X_2$ between two top. spaces $X_1, X_2$ we ~~can~~ define

$$t(f) : t(X_1) \longrightarrow t(X_2) \text{ by}$$

sending a closed subset to the closure of its image. (i.e $Y \longmapsto \overline{f(Y)}$, $Y \underset{\text{closed}}{\subset} X$)

This is well defined.

So, t is a functor on topological spaces.

Now, we can define a map

$$\alpha : X \longrightarrow t(X) \text{ by sending}$$

$$p \longmapsto \overline{\{p\}}$$

$\alpha$ gives a ~~b~~ bijection between the sets of open subsets of $X$ and the sets of open subsets of $t(X)$.

sets of $t(X) \setminus t(X \setminus U)$

the sch...

For an, $U \underset{\text{open}}{\subseteq} X$. then $\alpha(U) \equiv t(X) \cap \alpha(\ldots)$

Let $V$ be a variety over $k$ and $\mathcal{O}_V$ be its sheaf of regular functions.

We will show that
$$(t(V), \alpha_*(\mathcal{O}_V))$$ is a scheme

over $k$.

Note that, any variety can be covered by open affine subvarieties. It is sufficient to show that if $V$ is affine, then $(t(V), \alpha_*(\mathcal{O}_V))$ is a scheme.

Let $V$ be ~~the~~ a affine variety with affine co-ordinate ring $A$.

For a subset $Y \subseteq \mathbb{A}^n$, we define the ideal $I(Y)$ of $Y$ as-

$$I(Y) = \{ f \in k[x_1, \ldots, x_n] \mid f(P) = 0 \ \forall P \in Y \}$$

For an affine variety $Y$, the affine coordinate ring is defined as

$$A(Y) := k[x_1, \ldots, x_n] / I(Y)$$

We define a morphism of locally ringed spaces

$$\beta : (V, \mathcal{O}_V) \longrightarrow (X = \operatorname{spec} A, \underline{\mathcal{O}_X})$$

by sending $p \longmapsto \underline{m}_p$,

where $m_p$ is the ideal of $A$ consisting of all regular functions which vanish at $p$.

We can say that $\beta$ is a bijection of closed points of $V$ onto the set of $X$.

Also, $\beta$ is a homeomorphism onto its image.

<u>claim</u> :- To show $(t(V), \alpha_*(\mathcal{O}_V))$ is a scheme, it is enough to show that $\mathcal{O}_X(U) \cong \mathcal{O}_V(\beta^{-1}(U))$, for open subsets $U \subseteq X$.

$$\begin{tikzcd} V \arrow[r, "\alpha"] \arrow[dr, bend right, "\beta"'] & t(V) \arrow[d, "\gamma"] \\ & X = \operatorname{spec} A \end{tikzcd} \qquad \beta = \gamma \circ \alpha.$$

and $(X, \mathcal{O}_X) \cong (t(V), \alpha_*(\mathcal{O}_V))$

To show that, ...

we have to show, $\mathcal{O}_X \cong \gamma_X(\alpha_X(\mathcal{O}_V))$

For any open subset $U \subseteq X$

Define a map $\mathcal{O}_X(U) \longrightarrow \beta_* \mathcal{O}_V(U)$
$= \mathcal{O}_V(\beta^{-1}(U))$

by sending $s \longmapsto \underline{g}(P)$.

take, $p \in \beta^{-1}(U)$,

$$\mathcal{O}_X(U) \longrightarrow \mathcal{O}_{X,\beta(P)} \cong A_{m_p} \longrightarrow A_{m_p}/m_p \cong k$$

$$s \longrightarrow s(\beta(P)) \longrightarrow \frac{\bar{x}}{\bar{y}} \to \frac{\bar{x}(P)}{\bar{y}(P)} = g(P)$$

Also, $\underline{\mathcal{O}_X(U) \cong \mathcal{O}_V(\beta^{-1}(U))}$, for $U \subseteq X$ open.

$$\begin{tikzcd}
\mathcal{O}_X(U) \arrow[r] \arrow[d] & \mathcal{O}_V(\beta^{-1}(U)) \arrow[d] \\
\mathcal{O}_{X,q} \arrow[r] \arrow[d] & \mathcal{O}_{V,\beta^{-1}(q)} \arrow[d] \\
A_q \arrow[r] & A_q.
\end{tikzcd}$$

So, $(t(V), \alpha_X(\mathcal{O}_V)) \cong (X, \mathcal{O}_X)$

$t(V)$ is an affine scheme.

... morphism.

i.e., ... a morphism

Now, to give a morphism from $(t(V), \alpha_* \mathcal{O}_V) \longrightarrow \operatorname{Spec} k$.

it is enough to give a homo. ~~mor~~ of rings, namely

$$k \longrightarrow \Gamma(t(V), \alpha_*(\mathcal{O}_V)) = \Gamma(V, \mathcal{O}_V)$$

We send $\lambda \in k$ to the constant function $\lambda$ on $V$.

<u>HW</u>. Check that

$$\operatorname{Hom}_{Var(k)}(V, W) \longrightarrow \operatorname{Hom}_{Schs}(t(V), t(W))$$

is ~~bije~~ bijective.

This implies that $t$ is fully faithful.

Ex. $\alpha : V \longrightarrow t(V)$ induces a homeomorphism from $V$ onto the set of closed points of $t(V)$.

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