

Lecture 5:

Let us consider the polynomial ring $k[x_1, \dots, x_n]$, where k is an alg. closed field.

For a element $f \in k[x_1, \dots, x_n]$ we define the zero set of f as

$$Z(f) = \{ P \in \mathbb{A}^n \mid f(P) = 0 \}$$

For a subset $T \subseteq k[x_1, \dots, x_n]$ we define the zero set of T as

$$Z(T) = \{ P \in \mathbb{A}^n \mid f(P) = 0 \forall f \in T \}$$

▣ A subset $\gamma \subseteq \mathbb{A}^n$ is called an algebraic set if

$$\gamma = Z(T), \quad T \subseteq k[x_1, \dots, x_n]$$

We define the topology on \mathbb{A}^n by taking as closed sets subsets of the form $Z(T)$.

$$T \subseteq k[x_1, \dots, x_n].$$

An affine algebraic variety is an irreducible closed subset in \mathbb{A}^n .

A regular map:

... variety over

Let V be an affine variety over k . A map $f: V \rightarrow k$ is said to be regular at P if for each $P \in V$ there is a open nbhd U of P and elements $g, h \in k[x_1, \dots, x_n]$ s.t. h is nowhere zero on U & $f \equiv \frac{g}{h}$ on U .

A variety over k is an affine or quasi-affine or projective or quasi-projective variety.

An affine variety is an open subset of an affine variety.

Let S be the graded ring $k[x_0, \dots, x_n]$.

Then, for a homogenous polynomial $f \in S$, we define the zero set of f as.

$$Z(f) = \{ P \in \mathbb{P}^n \mid f(P) = 0 \}$$

For a subset T of homogenous elements in S , we define the zero set of T as

$$\mathbb{P}^n \cong \frac{\mathbb{A}^{n+1} \setminus \{0\}}{\sim}$$

$$(a_0, \dots, a_n)$$

$$\sim (\lambda a_0, \dots, \lambda a_n)$$

$$\lambda \neq 0$$

... T ?

$Z(U)$

$$Z(T) = \{ P \in \mathbb{P}^n \mid f(P) = 0 \quad \forall f \in T \}$$

III We define the topology on \mathbb{P}^n by taking as closed subset the sets of the form $Z(T)$, T is any set of homogenous elements in S .

IV A projective variety is an irreducible closed subset of \mathbb{P}^n .

V Let V be a projective variety over k . Given a map $f: V \rightarrow \mathbb{A}^1$ is regular if for each $P \in V$, there is a open nbd U of P and homogenous elements $g, h \in S$ of same degree s.t. h is nowhere 0 on U and $f = \frac{g}{h}$ on U .
A f is said to be regular on V if it is regular at each point P of V .

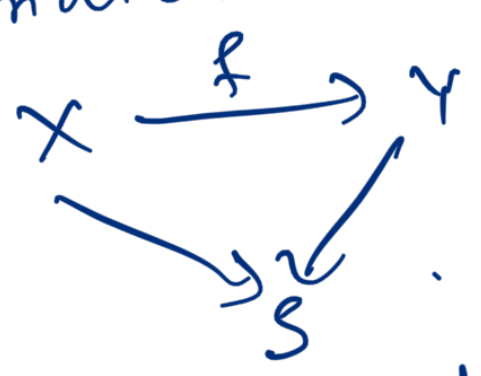
VI Now, let V be a variety over k . For an open subset $U \subseteq V$, let $\mathcal{O}(U)$ be the set of regular functions on U .
If $W \subseteq U$ the map

For open subset $U \subset V$, the natural restriction map $\rho_U: \mathcal{O}_V(U) \rightarrow \mathcal{O}_V(U)$ is the

restriction map. Given \mathcal{O} is a sheaf of rings on V , and is called the sheaf of regular functions on V .

Definition Let S be a fixed scheme. A scheme over S is a scheme X , together with the morphism $X \rightarrow S$.

For any two schemes X and Y over S , a morphism $f: X \rightarrow Y$ is said to be a S -morphism if it is compatible with the given morphisms to S . i.e., the following commutative diagram



We denote by Sch_S the category of schemes over S .

Theorem: - Let k be an alg. closed field.

Let k be a natural number.
 There is a natural faithful functor $t: \text{Var}(k) \rightarrow \text{Sch}_k$
 from the category of varieties over k to the category of schemes over k . For any variety V , its topological space is homeomorphic to the set of closed points of $\text{sp}(t(V))$ and its sheaf of regular functions is obtained by restricting the structure sheaf of $t(V)$ via this homeomorphism.

Proof :-

- Let X be any topological space.
 & let $t(X)$ be the set of all irreducible closed subsets of X .
- ▣ If γ is a closed subset of X then $t(\gamma) \subseteq t(X)$.
 - ▣ for two closed subsets γ_1 and γ_2 of X
 $t(\gamma_1 \cup \gamma_2) = t(\gamma_1) \cup t(\gamma_2)$.
 - ▣ for any family $\{\gamma_i\}$ of closed

VII
subsets of X .
 $t(\cap Y_i) = \cap t(Y_i)$

So, we define a topology on $t(X)$ by taking as closed sets, the subsets of the form $t(Y)$, where Y is a closed subset of X .

~~As~~ For, any cts map $f: X_1 \rightarrow X_2$ between two top. spaces X_1, X_2 we can define

$t(f): t(X_1) \rightarrow t(X_2)$ by sending a closed subset to the closure of its image, (i.e. $Y \mapsto \overline{f(Y)}$ where $Y \subseteq X$ closed)

This is well defined.

So, t is a functor on topological spaces.

Now, we can define a map $\alpha: X \rightarrow t(X)$ by sending $p \mapsto \{\overline{p}\}$

α gives a ~~bijection~~ bijection between the sets of open subsets of X and the sets of open subsets of $t(X)$.

For an, $U \subseteq X$ open, then $d(U) \cong \text{triv}$

Let V be a variety over k and \mathcal{O}_V be its sheaf of regular functions.

We will show that $(\text{triv}, d_*(\mathcal{O}_V))$ is a scheme

over k .

Note that, any variety can be covered by open affine subvarieties.

It is sufficient to show that if V is affine, then $(\text{triv}, d_*(\mathcal{O}_V))$ is a scheme.

Let V be a affine variety with affine co-ordinate ring A .

For a subset $\gamma \subseteq \mathbb{A}^n$, we define the ideal $I(\gamma)$ of γ as -

$$I(\gamma) = \{ f \in k[x_1, \dots, x_n] \mid f(P) = 0 \forall P \in \gamma \}$$

For an affine variety γ , the co-ordinate ring is defined as

$$A(\gamma) := k[x_1, \dots, x_n] / I(\gamma)$$

We define a morphism of locally ringed spaces

$$\beta: (V, \mathcal{O}_V) \longrightarrow (X = \text{Spec } A, \mathcal{O}_X)$$

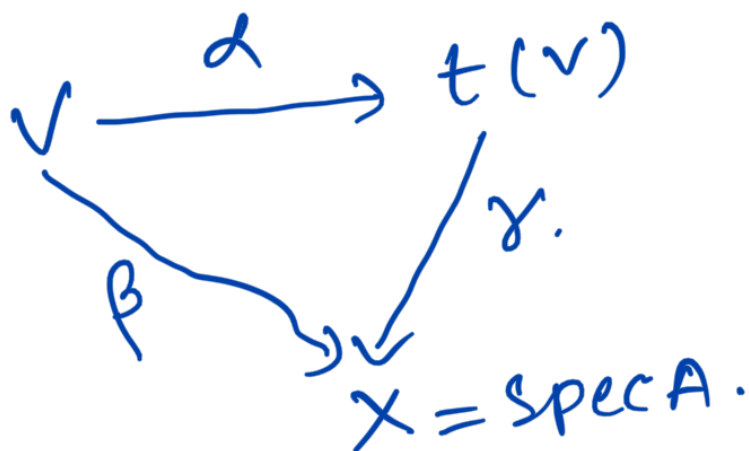
by sending $p \longmapsto \underline{m}_p$,

where \underline{m}_p is the ideal of A consisting of all regular functions which vanish at p .

We can say that β is a ~~1-1~~ bijection of V onto the set of closed points of X .

Also, β is a homeomorphism onto its image.

claim 0 - To show $(t(V), d_X(\mathcal{O}_V))$ is a scheme, it is enough to show that $\mathcal{O}_X(U) \cong \mathcal{O}_V(\beta^{-1}(U))$ for open subsets $U \subseteq X$.



$$\beta = \gamma \circ \alpha.$$

$$\dots \quad (X, \mathcal{O}_X) \cong (t(V), d_X(\mathcal{O}_V))$$

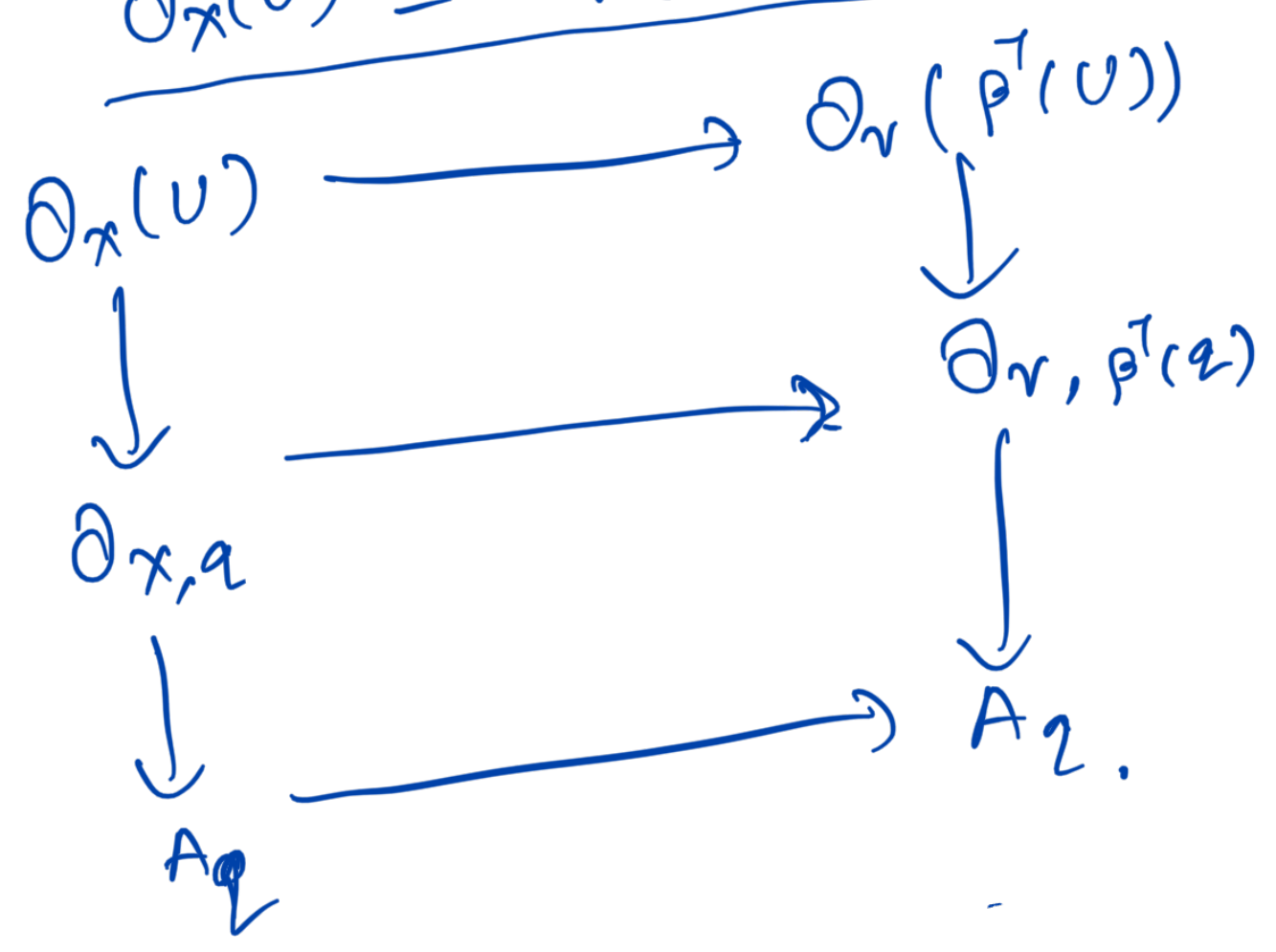
To show that $\mathcal{O}_X \cong \gamma_* (\mathcal{O}_V)$
 we have to show, $\mathcal{O}_X \cong \gamma_* (\mathcal{O}_V)$

For any open subset $U \subseteq X$
 Define a map $\mathcal{O}_X(U) \rightarrow \beta_* \mathcal{O}_V(U)$
 $= \mathcal{O}_V(\beta^{-1}(U))$

by sending $s \mapsto \underline{g(P)}$.

take, $p \in \beta^{-1}(U)$,
 $\mathcal{O}_X(U) \rightarrow \mathcal{O}_{X, \beta(p)} \cong A_{m, p} \rightarrow \frac{A_{m, p}}{m_p} \cong k$
 $s \mapsto \underline{s} \mapsto \frac{\underline{s}(p)}{\underline{s}(p)} = g(p)$

Also, $\mathcal{O}_X(U) \cong \mathcal{O}_V(\beta^{-1}(U))$, for $U \subseteq X$
 open.



So, $(t(V), d_X(\mathcal{O}_V)) \cong (X, \mathcal{O}_X)$
 $\therefore t(V)$ is an affine scheme.

Now, to give a morphism from $(t(V), d_X \mathcal{O}_V) \rightarrow \text{Spec } k$.
 it is enough to give a homo.
 $k \rightarrow \Gamma(t(V), d_X(\mathcal{O}_V)) = \Gamma(V, \mathcal{O}_V)$

We send $\lambda \in k$ to the constant function λ on V .

HW. Check that $\text{Hom}_{\text{Var}(k)}(V, W) \rightarrow \text{Hom}_{\text{Schs}}(t(V), t(W))$ is ~~be~~ bijective.

This implies that f is fully faithful.

Pr. $\alpha: V \rightarrow t(V)$ induces a homeomorphism from V onto the set of closed points of $t(V)$.