Construction of Proj:-

Graded ring:

Let S be a graded ring, together with a decomposition $S = \bigoplus_{d \geq 0} S_d$ of S into a direct sum of abelian groups S_d, s.t. for any $d, e \geq 0$

$$S_d \cdot S_e \subseteq S_{d+e}$$

Clearly S_0 is a subring, each S_d is an S_0-module and S is a S_0-algebra.

- An element of S_d is called a homogenous element of S of degree d.

- An ideal a of S is called a homogenous ideal if

$$a = \bigoplus_{d \geq 0} (a \cap S_d)$$

Some facts about homogenous ideal:

i) An ideal is homogenous iff it can be generated by homogenous elements.

ii) The sum, product, intersection and radical of homogenous ideals are homogenous.

iii) A homogenous ideal $a \subseteq S$ is prime iff for any two homogenous elements $f, g \in a$ $fg \in a \Rightarrow f \in a$ or $g \in a$..

Now. we define the set Proj S and then describe the topology and and structure sheaf on it.

Let. S_+ be the ideal $\bigoplus_{d>0} S_d$.

We define Proj S to be the set of all homogenous prime ideals P, s.t. $S_+ \nsubseteq P$.

If a is a homogenous prime ideal of S, we define

$\{ P \in \text{Proj } S \mid a \subseteq P \}$.

$V(a) = \{ p \in$ [illegible] $\}$

Lemma :-

(a) If a and b are homogenous ideals in S then $V(ab) = V(a) \cup V(b)$.

(b) If $\{a_i\}$ is any family of homogenous prime ideals of S then $V(\sum a_i) = \cap V(a_i)$

Proof :-

Proof is similar to the affine case. Just use iii) described above.

We define the topology on Proj S by taking the sets of the form $V(a)$ to be the closed subsets.

Now, we define the structure sheaf on Proj S.

Definition :-

For each $p \in$ Proj S, we define [illegible] the elements of [illegible]

$S_{(P)}$, to be the ... localized ring degree 0 in the multiplicative $T^{-1}S$, where T is a multiplicative subset of S, consisting of all homogenous elements not in P.

Now, for any open subset $U \subseteq \text{Proj}\, S$

we define $\mathcal{O}(U)$ to be the set of functions

$$s : U \longrightarrow \coprod_{P \in U} S_{(P)} \quad \text{s.t.}$$

$s(P) \in S_{(P)}$ for every $P \in U$

and, for each $P \in U$ there is a nbd V of P inside U, and homogenous elements a, f in S of same degree such that for all $q \in V$, $f \notin q$ & $s(q) = \frac{a}{f}$ in $S_{(q)}$.

It is clear that $\mathcal{O}$ is a pre-sheaf of rings with natural restrictions.

The local nature of definition ... that $\mathcal{O}$ is also a sheaf.

say,

Theorem:-

Let S be a graded ring.

(a) For any $p \in \operatorname{Proj} S$, the stalk $\mathcal{O}_p$ is isomorphic $S_{(p)}$.

(b) For any homogeneous $f \in S_+$ let $D_+(f) = \{p \in \operatorname{Proj} S \mid f \notin p\}$.

$D_+(f)$ is open in $\operatorname{Proj} S$ and for each such open set, we have an isomorphism of locally ringed spaces

$$(D_+(f), \mathcal{O}|_{D_+(f)}) \cong \operatorname{Spec} S_{(f)}.$$

where $S_{(f)}$ is the subring of elements of degree 0 in the localized ring ~~$S_{(f)}$~~ S_f.

(c) $\operatorname{Proj} S$ is a scheme.

Proof:-

(a) Note that we can write

(a) Note :
any element of $\mathcal{O}_p$ as $\langle U, s \rangle$ where U is a open set containing p & $s \in \mathcal{O}(U)$.

We define a map $\varphi : \mathcal{O}_p \longrightarrow S_{(P)}$ by sending s to s(P).
This is a well defined ring homomorphism.

<u>Surjectivity</u>:-
Any element of $S_{(P)}$ can be written as $\frac{a}{f}$, where a, f are homogenous elements of S of same degree.
Since, $f \notin P$, $P \in D_+(f)$
Then, $\frac{a}{f}$ defines a section of $\mathcal{O}(D_+(f))$ whose value at p is the given element.

<u>Injectivity</u>:-
... elements $\langle U, s \rangle$

Consider two ... such that $s(P) = t(P)$
& $\langle U_2, t \rangle$ such that

Now, shrinking the neighbour-hood of P if necessary
we may assume that for a open neighbourhood U of P
$$s = \frac{a}{f} \quad \& \quad t = \frac{b}{g} \text{ on } U.$$

where, a, b, f, g are homogenous elements of S, $f, g \notin P$
& a, f have same degree,
b, g have same degree.

Since, they have same image in $S_{(P)}$, $\exists$ a homogenous element $h \notin P$ s.t
$$h(ag - bf) = 0$$

Now, if we consider the open set $\bar{U} = D_+(h) \cap D_+(f) \cap D_+(g)$
then, $P \in \bar{U}$
and $s = t$ in $\bar{U}$.

(*) In the above proof of ...licity, if $\deg f = 0$

surjectivity.

Then, consider ~~the~~ a open neighbourhood $D_+(g_0 f)$ of P ~~s.t.~~ where g_0 has positive degree & $g_0 \notin P$. This is possible since $S_+ \notin P$.

(b) For proof see Hartshorne, Algebraic Geometry, Proposition 2.5 (b), Chapter 2.

Alternatively,

we can consider the inclusion

$$\operatorname{Spec} S_{(f)} \hookrightarrow \operatorname{Proj} S.$$

and on $D_+(f)$, take the structure sheaf of affine scheme $\operatorname{Spec} S_{(f)}$ as the structure sheaf of $D_+(f)$.

Then by glueing these structure sheaves over $D_+(f)$'s, for $f \in S_+$,

we can construct a

We ... Proj S
sheaf on Proj S
its restrictions to $D_+(f)$,
$f \in S_+$ ~~is~~. are the structure
sheaves of $\operatorname{Spec} S_{(f)}$, $f \in S_+$.

We call this sheaf on $\operatorname{Proj} S$,
its structure sheaf, $\mathcal{O}_{\operatorname{Proj} S}$.