LECTURE 3

Date of Lecture: September 28, 2021

The symbol $\mathcal{Ab}$ will denote the category of abelian groups. If X is a topological space, $\mathcal{Psh}_X$ and $\mathcal{Sh}_X$ denote the category of presheaves and the category of sheaves respectively on X. By a ring we mean a commutative ring with identity. The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

In this lecture will fix a topological space X, and τ will denote its collection of open sets.

1. **Ringed spaces**

1.1. **Local maps of local rings.** Let $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ be local rings. A *local homomorphism* or a *local map from A to B is a* ring homomorphism $\phi\colon A \to B$ such that $\phi(\mathfrak{m}_A) \subset \phi(\mathfrak{m}_B)$. This is equivalent to saying that $\phi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$. It is also equivalent to the condition that x is a unit in A whenever $f(x)$ is a unit in B.

1.1.1. The basic example of a local homomorphism is the following. Let $\phi\colon B \to A$ be a ring homomorphism, $\mathfrak{p}$ a prime ideal in A and $\mathfrak{q}$ the prime ideal in B given by $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$. Then the ring map $B_{\mathfrak{q}} \to A_{\mathfrak{p}}$ induced by ϕ, namely the composite

$$B_{\mathfrak{q}} \to A_{\mathfrak{q}} \to A_{\mathfrak{p}},$$

is a local homomorphism.

1.2. **The direct image of presheaves.** Let F be a presheaf on a topological space X. Let $f\colon X \to Y$ be a continuous map between topological spaces.The *direct image of F under f* is the presheaf f_*F on Y defined by the rule,

$$(f_*F)(U) = F(f^{-1}(U)), \tag{1.2.1}$$

as U varies over the open sets of Y. The restriction maps on f_*F are ones induced by the restriction maps of F.

In the above situation, if $x \in X$, then, by definition of direct limits, we have, for every open neighbourhood V of x, a map $\gamma_{V,x}\colon F(V) \to F_x$ such that the following diagram commutes

$$\begin{array}{ccc} F(V) & & \\ {\scriptstyle \rho^V_W}\downarrow & \searrow^{\gamma_{V,x}} & \\ F(W) & \xrightarrow[\gamma_{W,x}]{} & F_x \end{array}$$

for every open set W such that $x \in W \subset V$. Now suppose U is an open neighbourhood of $f(x)$ in Y. Then the map $\gamma_{f^{-1}(U),x}\colon F(f^{-1}(U)) \to F_x$ translates to a map $f_*F(U) \to F_x$. Moreover this map is compatible with open immersions $U' \subset U$

with $f(x) \in U'$, i.e., for such a pair U' and U the following diagram commutes:

$$\begin{array}{ccc} f_*F(U) & & \\ {\scriptstyle \rho^U_{U'}}\big\downarrow & \searrow & \\ f_*F(U') & \longrightarrow & F_x \end{array}$$

The universal property of direct limits then gives us an induced map of stalks:

$$(f_*F)_{f(x)} \longrightarrow F_x \tag{1.2.2}$$

1.2.3. Note that if F is a presheaf of rings, then so is f_*F and the map (1.2.2) is a ring homomorphism. Note also that if $F \in \mathcal{Sh}_X$ then $f_*F \in \mathcal{Sh}_Y$.

1.3. **Maps of Ringed spaces.** We defined a ringed space in first paragraph of § 1.4 of Lecture 1. Let $(X, \mathscr{O}_X)$ and $(Y, \mathscr{O}_Y)$ be ringed spaces. A *morphism of ringed spaced* $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ is a pair $(f, f^\#)$ with $f\colon X \to Y$ a continuous map and $f^\#\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$ a map of sheaves of rings. It is straightforward to see that with this definition of morphisms of ringed spaces, ringed spaces form a category.

Here are some examples maps of ringed spaces.

1. Let $f\colon X \to Y$ be a smooth map of smooth manifolds. The assignment $g \mapsto g \circ f$ gives us a ring homomorphism $\mathscr{C}^\infty_Y(U) \to \mathscr{C}^\infty_X(f^{-1}(U))$, which in turn defines a map of sheaves of rings (in fact a map of **R**-algebras) $\mathscr{C}^\infty_Y \to f_*\mathscr{C}^\infty_Y$. This makes f into a map of ringed spaces $(X, \mathscr{C}^\infty_X) \to (Y, \mathscr{C}^\infty_Y)$.
2. The same idea shows that a holomorphic map $f\colon X \to Y$ between complex manifolds is naturally a map of ringed spaces $(X, \mathscr{O}^{\rm hol}_X) \to (Y, \mathscr{O}^{\rm hol}_Y)$.
3. Let $\phi\colon B \to A$ be a ring homomorphism, $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and let $\mathscr{O}_X$ and $\mathscr{O}_Y$ the sheaves defined in §2.2 of Lecture 2 (see especially § 2.2.3 of *loc.cit.*). Let $f\colon X \to Y$ be the continuous map induced by ϕ, namely $\mathfrak{p} \mapsto \phi^{-1}\mathfrak{p}$, $\mathfrak{p}$ a prime ideal of A. If $g \in B$, then note that $f^{-1}(Y_g) = X_{\phi(g)}$ and the natural ring map $B_g \to A_{\phi(g)}$ translates to a ring map $f^\#(Y_g)\colon \mathscr{O}_Y(Y_g) \to \mathscr{O}_X(f^{-1}(Y_g))$. Using Problem 9 of HW1 we get a map of sheaves of rings $f^\#\colon \mathscr{O}_Y \to f_*\mathscr{O}_X$. The pair $(f, f^\#)$ is then a map of ringed spaces.

1.4. **Locally ringed spaces.** A *locally ringed space* is a ringed space $(X, \mathscr{O}_X)$ such that the stalks $\mathscr{O}_{X,x}$ are local rings for every $x \in X$. A *morphism of locally ringed spaces* is a map $(f, f^\#)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ of ringed spaces such that for each $x \in X$ the composite

$$\mathscr{O}_{Y,f(x)} \xrightarrow{\text{via } f^\#} (f_*\mathscr{O}_X)_{f(x)} \xrightarrow{(1.2.2)} \mathscr{O}_{X,x}$$

is a local homomorphism of local rings.

The examples in §1.3 are all maps of locally ringed spaces. In greater detail”

1. Let X be a smooth manifold. Then $(X, \mathscr{C}^\infty_X)$ is a locally ringed space. Indeed, let $x \in X$ and $\xi \in \mathscr{C}_{X,x}$. Then $\xi = s_x$ for some section s of $\mathscr{C}^\infty_X$ on an open neighbourhood U of x. Define the value $\xi(x)$ of ξ to be the value $s(x)$ of the smooth function s at x. One checks that $\xi(x)$ is well defined and $\xi \to \xi(x)$ gives a non-zero ring homomorphism $\mathscr{C}^\infty_{X,x} \to \mathbf{R}$. The kernel is a maximal ideal since $\mathbf{R}$ is a field. It is easy to see that it is the only maximal ideal of $\mathscr{C}^\infty_{X,x}$. Moreover (exercise!) if $f\colon X \to Y$ is a smooth map of smooth manifolds, then the induced map of ringed spaces described in §1.3 is a map of locally ringed spaces.

2. The same observations as in the above example hold for complex manifolds. If X is a complex manifold then $(X, \mathscr{O}_X^{\text{hol}})$ is a locally ringed space, and holomorphic maps between complex maniflds give rise to maps of locally ringed spaces.

3. Let A be a ring, $X = \operatorname{Spec} A$ and $x \in X$ a point. Then x is a prime ideal $\mathfrak{p}$ of A. We claim that $\mathscr{O}_{X,x} = A_{\mathfrak{p}}$. By definition, $\mathscr{O}_{X,x} = \varinjlim_{g \notin \mathfrak{p}} A_g$. It follows that if $g \notin \mathfrak{p}$, the image of $g \in A = \mathscr{O}_X(X)$ in $\mathscr{O}_{X,x}$ is a unit, since the map $A \to \mathscr{O}_{X,x}$ must factor through the localisation $A \to A_g$. By the universal property of localisations, we get a ring homomorphism $A_{\mathfrak{p}} \to \mathscr{O}_{X,x}$. On the other hand, since the natural ring maps $A_g \to A_{\mathfrak{p}}$ (namely $a/g^n \mapsto a/g^n$) give a map of directed systems, we get a a ring map $\mathscr{O}_{X,x} \to A_{\mathfrak{p}}$. These two processes are easily seen to be inverses of each other. Now suppose $\phi\colon B \to A$ is a ring homomorphism, $Y = \operatorname{Spec} B$ and $f\colon X \to Y$ the corresponding map of ringed spaces described in §1.3. Let $x \in X$. Then x is a prime ideal, say $\mathfrak{p}$, of A. Then $f(x)$ is the prime ideal $\mathfrak{q} = \phi^{-1}(\mathfrak{p})$. The map $\mathscr{O}_{Y,f(x)} \to \mathscr{O}_{X,x}$ described in (1.2.2) is easily seen to be the map in §1.1.1. In other words the map of ringed spaces $(X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y)$ described in Example 3 of §1.3 is a map of locally ringed spaces.

2. **Schemes**

2.1. **Definition of a scheme.** An *affine scheme* is a locally ringed space $(X, \mathscr{O}_X)$ isomorphic to $(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A})$ for some ring A. Note that in this case A is canonically isomorphic to $\mathscr{O}_X(X)$, and the affine scheme X determines the ring A up to canonical isomorphism.

A *scheme* is a locally ringed space $(X, \mathscr{O}_X)$ such that there is an open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ with $(U_\alpha, \mathscr{O}_X|_{U_\alpha})$ an affine scheme for each $\alpha \in \Lambda$.