# LECTURE 2

**Date of Lecture:** September 23, 2021

The symbol $\mathcal{A}b$ will denote the category of abelian groups. If $X$ is a topological space, $\mathcal{P}sh_X$ and $\mathcal{S}h_X$ denote the category of presheaves and the category of sheaves respectively on $X$. By a ring we mean a commutative ring with identity. The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

In this lecture will fix a topological space $X$, and $\tau$ will denote its collection of open sets.

## 1. **Functorial properties of sheafification**

Throughout this section $X$ is a topological space.

1.1. $(-)^+$ **as a functor.** Let $F, G \in \mathcal{P}sh_X$ and suppose $\varphi\colon F \to G$ is a morphism in $\mathcal{P}sh_X$. Then using the definition of the sheafification in (2.2.1) of Lecture 1 it is easy to see that one gets a map

$$\varphi^+\colon F^+ \longrightarrow G^+. \tag{1.1.1}$$

Indeed if $\sigma \in F^+(U)$ and $x$ a point in the open set $U$, then $\sigma(x)$ is the germ of some section $s$ of $F$ defined in a neighbourhood $V$ of $x$. Without loss of generality, we may assume $V \subset U$. Set $\sigma'(x) = (\varphi(V)(s))_x$. We leave it to the reader to verify that this gives a well-defined continuous map $\sigma'\colon U \to \mathscr{E}(G)$ such that $\pi\circ\sigma' = \mathbf{1}_U$. Define $\varphi^+(U)(\sigma) := \sigma'$. This gives $\varphi^+\colon F^+(U) \to G^+(U)$. We leave it you to check that $\varphi^+$ is a map of sheaves.

In the above situation it is not hard to see that the following diagram commutes:

$$\begin{array}{ccc} F & \xrightarrow{\ \theta_F\ } & F^+ \\ {\scriptstyle\varphi}\big\downarrow & & \big\downarrow{\scriptstyle\varphi^+} \\ G & \xrightarrow[\ \theta_G\ ]{} & G^+ \end{array} \tag{1.1.2}$$

The fancy way of saying this is that $\theta$ is a natural transformation from the identity functor on $\mathcal{P}sh_X$ to the sheafification functor on $\mathcal{P}sh$. More precisely, if $\mathbf{Fgt}\colon \mathcal{S}h_X \to \mathcal{P}sh_X$ is the forgetful functor which sends a sheaf to its underlying presheaf, then

$$\theta\colon \mathbf{1}_{\mathcal{P}sh_X} \to \mathbf{Fgt}\circ(-)^+.$$

1.2. **The universal property reinterpreted.** Towards the end of Lecture 1 we stated the universal property of sheafification. We give a few more details now. Suppose $F \in \mathcal{P}sh_X$ and $\mathscr{F} \in \mathcal{S}h_X$ and $\varphi\colon F \to \mathscr{F}$ is a map of presheaves. In a more pedantic fashion, the map is really $\varphi\colon F \to \mathbf{Fgt}(\mathscr{F})$ and that pedantry will be invoked soon. By (1.1.2) we have a commutative diagram in which the south pointing arrow on the right is an isomorphisn by Problem 3 of HW 1:

$$\begin{array}{ccc} F & \xrightarrow{\varphi} & \mathscr{F} \\ {\scriptstyle \theta_F}\downarrow & & \wr\downarrow{\scriptstyle \theta_{\mathscr{F}}} \\ F^+ & \xrightarrow[\varphi^+]{} & \mathscr{F}^+ \end{array}$$

Let $\tilde{\varphi}\colon F^+ \to \mathscr{F}$ be the bottom east pointing arrow followed by the inverse of the south pointing arrow on the right. In other words, for those who don't think in terms of commutative diagrams naturally, $\tilde{\varphi} = (\theta_{\mathscr{F}})^{-1} \circ \varphi^+$. It is immediate that $\varphi = \tilde{\varphi} \circ \theta_F$. The uniqueness of $\tilde{\varphi}$ is left to the reader. A hint is the following: Let $\psi\colon F^+ \to \mathscr{F}$ be any map of presheaves such that $\varphi = \psi \circ \circ \theta_F$. Since $F^+$ is a sheaf, $\theta_{F^+}$ is invertible. Now check that $\psi^+ = \varphi^+ \circ (\theta_{F^+})^{-1}$. How will you proceed from here?

The universal property can then be written as a bifunctorial isomorphism

(1.2.1) $$\mathrm{Hom}_{\mathscr{Psh}_X}(F,\, \mathbf{Fgt}\mathscr{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{Sh}_X}(F^+,\, \mathscr{F}).$$

In other words, the universal property is equivalent to saying $(-)^+$ is a left adjoint to **Fgt**.

## 2. $\mathscr{B}$-presheaves

In what follows $X$ is a topological space and $\mathscr{B}$ a base for the topology on $X$. We define a category $\mathscr{C}_{\mathscr{B}}$ as follows. The objects are members of $\mathscr{B}$ and the morphisms are given by

$$\mathrm{Hom}_{\mathscr{C}_{\mathscr{B}}}(U,\, V) = \begin{cases} \emptyset & \text{if } U \text{ is not a subset of } V \\ U \subset V & \text{otherwise.} \end{cases}$$

If $(B_\alpha)$ is a family of members of $\mathscr{B}$, we write $B_{\alpha\beta}$ for the intersection $B_\alpha \cap B_\beta$.

2.1. **Definitions.** We are primarily interested in the case where the intersection of two members of $\mathscr{B}$ is again a member of $\mathscr{B}$. Fix such a base $\mathscr{B}$. A $\mathscr{B}$-presheaf $F$ on $X$ is an $\mathcal{Ab}$-valued contravariant functor on $\mathscr{C}_{\mathscr{B}}$. In other words $F$ can be written in a covariant way as

$$F\colon \mathscr{C}_{\mathscr{B}}^{\circ} \longrightarrow \mathcal{Ab}.$$

A $\mathscr{B}$-sheaf $\mathscr{F}$ is an $\mathscr{B}$-presheaf with the following additional property. Let $B \in \mathscr{B}$ and $(B_\alpha)_{\alpha\in\Lambda}$ be a cover of $B$ by members of $\mathscr{B}$ and suppose we have elements $s_\alpha \in \mathscr{F}(B_\alpha)$, $\alpha \in \Lambda$ such that $s_\alpha|_{B_{\alpha\beta}} = s_\beta|_{B_{\alpha\beta}}$ for all $\alpha$ and $\beta$ in $\Lambda$. Then there exists a unique section $s \in \mathscr{F}(B)$ such that $s|_{B_\alpha} = s_\alpha$ for every $\alpha \in \Lambda$. Note that we have combined the usual two conditions in the definition of a sheaf into one condition by insisting on uniqueness.

**Remark 2.1.1.** One can generalise these definitions for a general base $\mathscr{B}$, i.e. one that is not necessarily closed under intersections. The definition of a $\mathscr{B}$-presheaf is as above. However greater care is needed in defining a $\mathscr{B}$-sheaf. So as before let $B \in \mathscr{B}$ and $(B_\alpha)_{\alpha\in\Lambda}$ a cover of $B$ by members of $\mathscr{B}$. Suppose for every pair of indices $\alpha$ and $\beta$ the collection $(B_{\alpha\beta k})_k$, $k$ in some index set $\Gamma_{\alpha\beta}$, is a cover of $B_{\alpha\beta}$ by members of $\mathscr{B}$. Then a $\mathscr{B}$-presheaf $\mathscr{F}$ is a $\mathscr{B}$-sheaf if the following is true: If $s_\alpha \in \mathscr{F}(B_\alpha)$, $\alpha \in \Lambda$, are sections such that $s_\alpha|_{B_{\alpha\beta k}} = s_\beta|_{B_{\alpha\beta k}}$ for all triples

$(\alpha,\, \beta,\, k)$, with $\alpha, \beta \in \Lambda$ and $k \in \Gamma_{\alpha\beta}$, then there exists a unique section $s \in \mathscr{F}(B)$ such that $s|_{B_\alpha} = s_\alpha$ for all $\alpha \in \Lambda$. It is not hard to see that in the event $\mathscr{B}$ is closed under intersection then this definition of a $\mathscr{B}$-sheaf is equivalent to what was given earlier.

2.2. **A $\mathscr{B}$-sheaf on the prime spectrum of a commutative ring.** Let $A$ be a ring and $X = \operatorname{Spec} A$ the prime spectrum of $A$. For any ideal $I$ of $A$, $V(I)$ is the set of prime ideals of $A$ containing $I$. Recall that the topology on $X$ is defined by decreeing sets of the form $V(I)$ as closed sets. Further, $V(I) = V(J)$ if and only if $\sqrt{I} = \sqrt{J}$. For $f \in A$, the complement of $V(\langle f \rangle)$ in $X$ is denoted $X_f$, and sets of the form $X_f$ form a base $\mathscr{B}$ for the topology on $X$. It is well known that $X_f$ can be identified with $\operatorname{Spec} A_f$ where $A_f$ is the localisation of $A$ with respective to the multiplicative system $\{1, f, f^2, \ldots, f^n, \ldots\}$. Note that $X_f \cap X_g = X_{fg}$ for $f, g \in A$, and hence $\mathscr{B}$ is closed under intersections.

If $X_f \subset X_g$, then the image $g/1$ of $g$ in $A_f$ is not contained in any prime ideal of $A_f$ and hence is a unit in $A_f$. This gives a natural map $A_g \to A_f$ of rings by the universal property of localisations. If $X_f = X_g$, one checks easily that the resulting maps $A_f \to A_g$ and $A_g \to A_f$ are inverses of each other, and hence $A_f$ can be canonically identified with $A_g$. In more down to earth terms, the fraction $f/1$ in $A_g$ is invertible, and hence $(f/1)^{-1} = b/g^d$ for some $b \in A$ and some $n \geq 0$. Then an element $a/f^n$ in $A_f$ can be identified with the element $(ab^n)/g^{nd}$ in $A_g$.

Define a $\mathscr{B}$-preseheaf of rings $F$ on $X$ by the formula

$$F(X_f) = A_f.$$

As we mentioned above, if $X_f \subset X_g$, we get a canonical map of rings $A_g \to A_f$, and this defines our restriction map $\rho^{X_g}_{X_f} \colon F(X_f) \to F(X_g)$. Since the process of taking localisations is transitive, this data makes $F$ into a $\mathscr{B}$-presheaf of rings on $X$. Note that

$$F(X) = A.$$

In fact $F$ is a $\mathscr{B}$-sheaf. To see this suppose $\mathscr{U} = (X_{f_\alpha})_{f\alpha \in \Lambda}$ is a cover of $X$ by members of $\mathscr{B}$ and $s \in A$ is such that $s|_{X_{f_\alpha}} = 0$ for every $\alpha \in \Lambda$. We claim that $s = 0$. Since $\cup_{\alpha \in \Lambda} X_{f_\alpha} = X$, if $\mathfrak{p} \in \operatorname{Spec} A$, then some $f_\alpha \notin \mathfrak{p}$. This means that the $I$ generated by the $f_\alpha$ is the unit ideal $A$. Hence there exist $\alpha_1, \ldots, \alpha_d \in \Lambda$ such that $\cup_{i=1}^d X_{f_{\alpha_i}} = X$. The condition $s|_{X_{\alpha_i}} = 0$ for $i = 1, \ldots, d$, amounts to saying that the image $s/1$ of $s$ in $A_{f_i}$ is zero for $i = 1, \ldots, d$. This in turn means that there exist positive integers $m_i$ such that $f_{\alpha_i}^{m_i} s = 0$ for $i = 1, \ldots, d$. Let $g_i = f_{\alpha_i}^{m_i}$. Then $g_i s = 0$ for every $i$. Since $X_{f_{\alpha_i}} = X_{g_i}$, we have $\cup_{i=1}^d X_{g_i} = X$, whence, arguing as before, the ideal $\langle g_1, \ldots, g_d \rangle$ generated by the $g_i$ is the unit ideal $A$. In other words, there are elements $b_1, \ldots, b_d \in A$ such that $\sum_{i=1}^d b_i g_i = 1$. It follows that

$$s = 1 \cdot s = \left( \sum_{i=1}^{d} b_i g_i \right) s = 0.$$

To simplify notation, write $U_\alpha = X_{f_\alpha}$ and $V_i = X_{g_i} (= X_{f_{\alpha_i}})$. Write $U_{\alpha\beta}$ and $V_{ij}$ for the intersections $U_\alpha \cap U_\beta$ and $V_i \cap V_j$. Now suppose we have elements $s_\alpha \in F(U_\alpha)$, $\alpha \in \Lambda$ such that $s_\alpha|_{U_{\alpha\beta}} = s_\beta|_{U_{\alpha\beta}}$. Let $s_{\alpha_i} = a_i/g^{n_i}$, $i = 1, \ldots, d$. Since $A_{g_i^{n_i}} = A_{g_i}$, replacing $g_i^{n_i}$ by $g_i$ if necessary, we write $s_{\alpha_i} = a_i/g_i$. The condition $s_{\alpha_i}|_{V_{ij}} = s_{\alpha_j}|_{V_{ij}}$ is equivalent to the saying there exist positive integers

$m_{ij}$ such that $(g_ig_j)^{m_{ij}}(g_ja_i - g_ia_j) = 0$. Let $n = \max(m_{ij})$. Then we have

$$g_i^{n+1}g_j^n a_j = g_j^{n+1}g_i^n a_i, \qquad i, j = 1, \ldots, d. \tag{2.2.1}$$

Since $V_i = \mathrm{Spec} A_{g_i^{n+1}}$, and since $\cup_{i=1}^d V_i = X$, arguing as before, we can find $r_1, \ldots, r_d \in A$ such that

$$\sum_{i=1}^{d} r_i g_i^{n+1} = 1. \tag{2.2.2}$$

Fix $j \in \{1, \ldots, d\}$. Multiplying both sides of (2.2.2) by $g_j^n a_j$ and using (2.2.1) we get

$$g_j^{n+1} \sum_{i=1}^{d} r_i g_i^n a_i = g_j^n a_j.$$

Let $s = \sum_{i=1}^d r_i g_i^n a_i$. Then the above equation shows that the image of $s$ in $A_{g_j}$ is $a_j/g_j$. In other words $s|_{V_j} = s_{\alpha_j}$ for $j = 1, \ldots, d$.

We claim $s|_{U_\alpha} = s_\alpha$ for all $\alpha \in \Lambda$. Fix $\alpha \in \Lambda$. Set $W_i = V_i \cap U_\alpha$, $i = 1, \ldots, d$. Since $V_i = U_{\alpha_i}$, $W_i = U_{\alpha_i} \cap U_\alpha$ and hence

$$s_{\alpha_i}|W_i = s_\alpha|_{W_i} \qquad i = 1, \ldots, d.$$

Now

$$(s|_{U_\alpha})|_{W_i} = s|_{W_i} = (s|_{V_i})|_{W_i} = s_{\alpha_i}|_{W_i} = s_\alpha|_{W_i} \qquad i = 1, \ldots, d.$$

Thus $(s|_{U_\alpha} - s_\alpha)|_{W_i} = 0$ for $i = 1, \ldots, d$. Since $\{W_i \mid i = 1, \ldots, d\}$ is an open cover of $U_\alpha$ by members of $\mathscr{B}$, replacing $A$ by $A_{f_\alpha}$ in our earlier argument, we see that $s|_{U_\alpha} = s_\alpha$.

To show $F$ is a $\mathscr{B}$-sheaf, we have to consider an arbitrary member $X_f$ of $\mathscr{B}$ and verify the $\mathscr{B}$-sheaf conditions for a cover $(X_{f_\alpha})$ of $X_f$. However, if $g_\alpha$ is the image of $f_\alpha$ in $A_f$, then identifying $X_f$ with $\mathrm{Spec}\, A_f$ we see that $X_{f_\alpha}$ gets identified with $(X_f)_{g_\alpha}$. The arguments we used above for $X$ and $A$ then apply without change to $X_f$ and $A_f$ and we see that $F$ is a $\mathscr{B}$-sheaf.

**2.2.3.** From Problem 8 of HW 1 we see that $F$ gives rise to a unique sheaf of rings in $X$ whose restriction to $\mathscr{B}$ is $F$. This sheaf is usually denoted $\mathscr{O}_X$ or $\mathscr{O}_{\mathrm{Spec}\, A}$. Thus $F = \mathscr{O}_X|_{\mathscr{B}}$. The ringed space $(X, \mathscr{O}_X)$ is called an *affine scheme.* A scheme is a ringed space which is locally an affine scheme. We will do this in greater detail later.