Lecture 18:

Cartier Divisors :-

Let X be a scheme.
For an open affine subset $U = \operatorname{spec} A$ of X, let S be the set of elements of A which are non zero divisors in A. Then, S is a multiplicative subset of A.

Let, $K(U)$ be the localisation of A by multiplicative subset S.
We call $K(U)$, the <u>total quotient ring</u> of A.

For any open subset U of X,

$$S(U) := \left\{ s \in \Gamma(U, \mathcal{O}_X) \text{ s.t. } s \text{ is not a zero divisor in any of the local ring } \mathcal{O}_{X,x},\ x \in U. \right\}$$

Then, $S(U)$ is a multiplicative subset of $\Gamma(U, \mathcal{O}_X)$.

Then the rings

$$S(U)^{-1}\Gamma(U, \mathcal{O}_X) \text{ form}$$

a presheaf.
and we call the associated ... the

$s_1, s_2 \in S(U)$

$s_1 s_2$

$\frac{s}{t} \in S(U)^{-1}\Gamma(U, \mathcal{O}_X)$

S

$\frac{s}{t} \to \frac{s|_V}{t|_V}$

sheaf of rings. The sheaf of total quotient rings of $\mathcal{O}$ and we denote it by $\mathcal{K}$.

We denote by $\mathcal{K}^*$ the sheaf of multiplicative groups of invertible elements in $\mathcal{K}$. Similarly we denote by $\mathcal{O}^*$ the sheaf of invertible elements of $\mathcal{O}$.

Cartier divisor:

A cartier divisor on a scheme X is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}^*$.

We can see that a ~~catier~~ cartier divisor of X can be expressed by giving a open cover $\{U_i\}$ of X and for each i, an element $f_i \in \Gamma(U_i, \mathcal{K}^*)$ s.t. $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^*)$ for all i, j.

A cartier divisor is principal if it is in the image of the map

$$\Gamma(X, \mathcal{K}^*) \longrightarrow \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$$

Two cartier divisors are linearly equiv. if their difference is a principal divisor.

$s \in \Gamma(X, \mathcal{K}^*/\mathcal{O}^*)$

$x \in X$, $s_x \in \mathcal{K}_x^*/\mathcal{O}_x^*$.

$s_x = \bar{f}_x$

$f_x \in \mathcal{K}_x^*$

$U_x \ni x$.

$f_x = \langle U_x, f_i \rangle$

$\langle U_x, s_i \rangle$

$s_i \in \mathcal{K}(U_x)$

$s_i = \bar{f}_i$ in $\mathcal{K}(U_x)/\mathcal{O}(U_x)$

We denote the group of cartier divisors modulo linear. equiv. by $\mathrm{CaCl}(X)$.

Proposition:-

Let X be an integral, separated noetherian scheme whose local rings are UFD. Then.

$$\mathrm{Cl}(X) \cong \mathrm{CaCl}(X).$$

Proof:-

First note that, since X is integral.

... stant sheaf

$\mathcal{K}^*$ is the constant ... associated to the function field $K(X)$ of X.

Consider a cartier divisor $\{(U_i, f_i)\}$.

Now, $f_i \in \Gamma(U_i, \mathcal{K}^*) = K(X)^*$

For each prime divisor Y. For co-efficient of Y to take the be, $v_Y(f_i)$.

Now, since $\frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j)$

$v_Y(f_i) = v_Y(f_j)$

v_Y is the discrete valuation to the DVR $\mathcal{O}_{X,\eta}$ where $\overline{Y = \{\eta\}}$

Then we have a Weil divisor,

$$D = \sum v_Y(f_i) Y_i \text{ on } X.$$

Conversely:-

Let D be a Weil divisor on X.

Then, for any $x \in X$, ~~we can~~ D induces a Weil divisor

D_x in $\operatorname{spec} \mathcal{O}_x$.

If $D = \sum n_i Y_i$.

$$D_x = \sum_{x \in Y_i} n_i Y_i$$

Now, $\mathcal{O}_x$ is a UFD.

Then, $\operatorname{cl}(D_x) = 0$.

i.e, $D_x = (f_x)$, $\underline{f_x \in K(X)}$.

For a the principal (f_x) in X,

$$(f_x)|_{\operatorname{spec} \mathcal{O}_x} = D|_{\operatorname{spec} \mathcal{O}_x} = D_x.$$

So, (f_x) & D differs only at prime divisors Y where $x \notin Y$. There are finitely many of them with non-zero coefficient in D or (f_x).

So, there is a open nbd $\underline{U_x}$ of x. s.t. $D|_{U_x} = (f_x)|_{U_x}$.

Now, we claim that $\{(U_x, f_x)\}_{x \in X}$ forms a

x | in $U_x \cap U_y$

Cartier divisor on X.

$(f_x)_{|U_x \cap U_y} = (f_y)_{|U_x \cap U_y}$.

An invertible sheaf on a scheme X is defined to be a locally free $\mathcal{O}_X$-module of rank 1.

Prop:-

If $\mathcal{L}$ & $\mathcal{M}$ are invertible sheaves on a scheme X, ~~Then~~, then, $\mathcal{L} \otimes \mathcal{M}$ is also invertible.

and $\exists$ an invertible sheaf $\mathcal{L}^{-1}$ on X s.t. $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.

Proof

~~Obvio~~ If for an open subset $U \subseteq X$

$\mathcal{L}_{|U} \cong \mathcal{O}_{X|U} \cong \mathcal{M}_{|U}$.

Then, $\mathcal{L}_{|U} \otimes \mathcal{M}_{|U} \cong (\mathcal{L} \otimes \mathcal{M})_{|U}$

$\cong \mathcal{O}_{X|U}$.

Form the presheaf $U \mapsto \mathcal{L}(U) \otimes \mathcal{M}(U)$

We take, $\mathcal{L}'$ to be,
the dual sheaf $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$.

Then,
$\mathcal{L}^\vee \otimes \mathcal{L} = \mathcal{O}_X$. (HW).

$U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$

$\mathcal{F}, \mathcal{G}$ - $\mathcal{O}_X$-module

$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ is a sheaf of $\mathcal{O}_X$-module)

For any scheme X. we define the Picard group of X, Pic X, to be the group of isomorphism classes of invertible sheaves under $\otimes$.

claim:-

On any scheme X. we
$\mathrm{CaCl}(X) \hookrightarrow \mathrm{Pic}(X)$

If X is integral, then,
$\mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$.

Def:-

Let D be a cartier divisor on a scheme X, ~~described by~~ represented by $\{(U_i, f_i)\}$.
... subsheaf $\mathcal{L}(D)$

We define a ... of $\mathcal{K}$. (sheaf of total quotient ring of X).

s.t. $\mathcal{L}(D)_{|U_i} = \langle f_i^{-1} \rangle$ as

an $\mathcal{O}_{U_i}$-module.

We call $\mathcal{L}(D)$, the sheaf associated to D.

Prop:-

Let X be a scheme (a) Then map $D \longmapsto \mathcal{L}(D)$ gives a 1-1 correspondence between Cartier divisors on X & invertible subsheaves of $\mathcal{K}$.

(b) $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$

(c) $D_1 \sim D_2 \iff \mathcal{L}(D_1) \cong \mathcal{L}(D_2)$.

as (as abstract invertible sheaves)

Proof:-

Let D is given by $\{(U_i, f_i)\}$

a

a Then $\mathcal{L}(D)_{|U_i} \cong \langle f_i^{-1} \rangle$

Conversely, ~~$\mathcal{L}(D)$~~ for any ~~inversely~~ invertible subsheaf $\mathcal{L}$ of $\mathcal{K}$. there is a open cover $\{U_i\}$ of X. s.t.

$\mathcal{O}_{U_i} \cong \mathcal{L}_{|U_i}$.

~~Let~~. Take the images of 1 ~~in~~ of $\mathcal{O}(U_i)$ in $\mathcal{L}(U_i)$.

(c) Its enough to show that if D is principal. then.

$$\mathcal{L}(D) \cong \mathcal{O}_X.$$

Since, D is given by $f \in \Gamma(X, \mathcal{K}^*)$ $\mathcal{L}(D)$ is globally generated by f^{-1}.

Prop:- Let X be a integral scheme. Then, ~~Cacl~~ the homomorphism

$\mathrm{CaCl}(X) \longrightarrow \mathrm{Pic}(X)$ is an isomorphism.

Proof

Consider ~~the~~ $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{K}$.

Since, $\mathcal{K}$ is the constant sheaf associated to $K(X)$.

$$\mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}.$$

and the natural map

$$\mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$$

expresses $\mathcal{L}$ as a subsheaf of $\mathcal{K}$.

Coro:

If X is noetherian, integral, separated scheme s.t. $\mathcal{O}_x$'s are UFD for all $x \in X$.

Then $\boxed{Cl(X) \cong CaCl(X) \cong \operatorname{Pic} X}$.

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