Weil Divisors :-

- **Regular local ring :-**

Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and the residue field $k = A/\mathfrak{m}$. Then A is said to be a regular local ring if

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A.$$

- A scheme X is regular in codimension one if every local ring $\mathcal{O}_x$ of X of dimension 1 is regular.

Example 1 (

Non-singular varieties.

Example 2.

Noetherian normal schemes.

- A scheme X is normal if every local ring $\mathcal{O}_{X,x}$ of X is integrally closed.

Prop (1) :- (A.M. - Pag 94) ✓

A be a noetherian local

Let A be ... noetherian local
domain of dimension 1. TFAE.
i) A is integrally closed.
ii) A is a regular local ring
iii) A is discrete valuation ring
iv) $\mathfrak{m}$ is a principal ideal
($\mathfrak{m}$ - max. ideal of A).

- X is a noetherian integral separated scheme which is regular in co-dimension 1. ---- $(*)$

Let X satisfies $(*)$.
A <u>prime divisor</u> on X is a closed integral subscheme Y of codimension one.

A <u>Weil divisor</u> is an element of the free abelian group $\mathrm{Div}(X)$, generated by prime divisors.

So, we can write a Weil divisor D as

$\sum n_i Y_i$

$D = \sum \ldots$

where, Y_i are prime divisors on X
n_i are integers & only finitely many are non-zero.

▨ If $n_i \geq 0 \ \forall i$ then D is said to be effective.

▨ Let Y be a prime divisor.
& η be its generic point, i.e.,
$Y = \overline{\{\eta\}}$

If Y is a closed irr. subset of a scheme X, then
$$\text{codim}(Y, X) = \dim(\mathcal{O}_{X,\eta})$$
Since, Y is a prime divisor.
$$\text{codimension}(Y, X) = 1 = \dim(\mathcal{O}_{X,\eta})$$

From Prop(1), above.
we get $\mathcal{O}_{X,\eta}$ is a DVR.

▨ **Function field $K(X)$ of a scheme X.**

Let X be an integral scheme. ... is the

Let X ... where η ...
Then, $\mathcal{O}_{X,\eta}$, where η is ..., is a field ... generic point of X.
This is called the function field of X, and denoted by $K(X)$.

Note that, If $X = \operatorname{Spec} A$, then, $K(X)$ is the quotient field of A.

- We can see that then quotient field of $\mathcal{O}_{X,\eta}$ is $K(X)$.

We call the corresponding discrete valuation v_Y the valuation of Y.

Now, since <u>X is separated</u>, Y is uniquely determined by its valuation.

Let $f \in K(X)^*$ be any non-zero rational function on X.
Then $v_Y(f) \in \mathbb{Z}$.

<u>Lemma</u>:-

Let X satisfies (*). & let $f \in K(X)$

Then $v_Y(f) = 0$ for all but finitely many prime divisors of X.

<u>Proof</u>:-

$f \in K(X)^* = (\mathcal{O}_{X,\xi})^*$.

So, we can choose an affine open subset $U (= \operatorname{Spec} A)$ of X s.t. $f \in \mathcal{O}_X(U)$.

Then, $Z = X - U$ is a proper closed subset of X.

Now, if Codimension $Z \geq 2$, then Z doesn't contain any prime divisor of X.

If Co-dim $Z = 1$, then prime divisors contained in Z are irr. ~~cop~~ components.

Since X is noetherian Z can contain at most finitely many prime divisors of X.

That implies that all the other ... must intersect U.

... prime divisors ...

It is sufficient to show that there are finitely many prime divisors of Y of U s.t. $v_Y(f) \neq 0$.

Y - p.d. in X
$\downarrow$
$Y \cap U$ p.d. in U

Since, $\mathcal{O}_{X,\eta}$ is DVR, $\mathfrak{m}_\eta$ - max. ideal of $\mathcal{O}_{X,\eta}$ is a principal ideal.

Let, $\mathfrak{m}_\eta = (m)$.

Then, any element $g \in \mathcal{O}_{X,\eta}$ can be written uniquely as

$$g = u m^k,$$

where u - is a unit in $\mathcal{O}_{X,\eta}$ & $k \geq 0$.

We can define a map from

$$\mathcal{O}_{X,\eta} \setminus \{0\} \longrightarrow \mathbb{Z}$$
$$g \longmapsto k.$$

by

Now, then function field $K(X)$ is quotient field of $(A_\eta = \mathcal{O}_{X,\eta})$.

So, for any element $\frac{g}{h} \in K(X)^*$, we have, g, h are in $\mathcal{O}_{X,\eta}$, we have, ... k_2

$g = u_1 m^{k_1}, \quad h = u_2 m^{k_2}$.

u_1, u_2 - units in $\mathcal{O}_{X,\eta}$ & $k_1, k_2 \geq 0$.

Then, we can extend the above map to

$$\mathrm{Frac}(\mathcal{O}_{X,\eta}) \setminus \{0\} \longrightarrow \mathbb{Z}$$

by $$\frac{g}{h} \longmapsto k_1 - k_2.$$

Since, Y is uniquely determined by its valuation,

we get that
$$v_Y(f) \geq 0.$$

Now, $v_Y(f) > 0 \Leftrightarrow Y \subseteq V(f)$

$\langle f \rangle$ is ~~co~~ an ideal in A.

$v_Y(f) > 0$, only when,
$$f \in \underline{m_\eta} = PA_P, \quad P\text{-prime in } A.$$

where $Y = V(P) = \{P\}$

$$\Rightarrow Y = V(P) \subseteq V(f).$$

But $V(f)$ is a proper closed subset of X, hence $V(f)$ can contain only finitely many divisors of X.

prime divisor.

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Def:-
Let X satisfy $(*)$ & $f \in K(X)^*$. We define the divisor of f, denoted by (f), as

$$(f) = \sum v_{Y_i}(f) \cdot Y_i$$

Such a divisor is called a principal divisor of X.

☒ There is a homomorphism

$$K(X)^* \longrightarrow \mathrm{Div}(X)$$

defined by

$$f \longmapsto (f).$$

Def:-
Two divisors D & D' are said to be linearly equivalent, denoted by $D \sim D'$ if $D - D' = (f)$ for some $f \in K(X)^*$.

The group $\mathrm{Div}(X)$ divided by the subgroup of principal divisors ... divisor the ... the divisor divisors. is called the divisor class group, $Cl(X)$.

Question: $Cl(\mathbb{A}^n_k) = ?$

$$Cl(\mathbb{A}^n_k) = 0.$$

This is given by a prop. which says that

For a noetherian domain A,

A is UFD $\iff$ $X = \operatorname{spec} A$ is normal & $Cl(X) = 0$.

See. H. Prop. 6.2. (II).

Question: $Cl(\mathbb{P}^n_k) = ?$

Theorem:

Let $X = \mathbb{P}^n_k$. k - alg closed field.
For any divisor $D = \sum n_i Y_i$
we define $\deg D$ as.

$$\deg D = \sum n_i \deg Y_i,$$

where, $\deg Y_i$ is the degree of the hypersurface Y_i.

... closed irr. pdy.

$V(f)$, f - closed ...

Let, $H = \underline{V(x_0)}$, $\mathbb{P}^n_k = \text{Proj}\, k[x_0, \ldots, x_n]$
hyperplane in $\mathbb{P}^n$.

Thm.

(a) If D is any divisor of degree d, then $D \sim dH$.

(b) for any $f \in K^*$, $\deg(f) = 0$

(c) The degree function gives an isomorphism.

$$\deg : Cl(X) \longrightarrow \mathbb{Z}$$

<u>Proof</u>:-

Let $S = k[x_0, \ldots, x_n]$ be the homo. co-ordinate ring of $\mathbb{P}^n_k$.

If g is homo. of deg d.

then, $g = g_1^{m_1} \cdots g_r^{m_r}$.

$$\Rightarrow (g) = \sum n_i Y_i, \quad Y_i = V(g_i), \; i = 1, \ldots, r.$$

Now, any $\underline{f \in K(X)^*}$ can be written as $f = \frac{g}{h}$, where

g, h are homogenous polynomial degree :

of same deg.

Then $(f) = (g) - (h)$.

$\Rightarrow \deg(f) = \deg(g) - \deg(h)$

$= 0$. — proves (b)

For (a)

Note that any divisor D of deg d. can be written as

$$D = D_1 - D_2$$

where D_1, D_2 are effective divisors of degree d_1, d_2, res.

Then, $d = d_1 - d_2$.

and $D_1 = (g_1)$, $D_2 = (g_2)$.

Then, $D - dH$

$= D_1 - D_2 - dH$

$= \left(\frac{g_1}{g_2 x_0^d}\right)$.

$\Rightarrow D \sim dH$.

suppose $D_1 = \sum n_i Y_{1i}$.

$Y_{1i} = V(g_{1i})$.

$D_1 = (g_1)$

$g_1 = g_{1i}^{n_1} \cdots g_{1r}^{n_r}$

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Proposition:-

Let X satisfy (*). & let Z be
... closed subset of X.

a proper ... Then.
& let $U = X - Z$. Then.

(a) ~~The~~ there is a surjective homo. $Cl(X) \longrightarrow Cl(U)$. Defined by $D = \sum n_i Y_i \longmapsto \sum n_i (Y_i \cap U)$.

(b) If $\underline{\operatorname{codim}(Z, X) \geq 2}$. then $Cl(X) \cong Cl(U)$.

(c) If Z is irr. & $\operatorname{codim}(Z,X) = 1$. then there is an exact seq.

$$\mathbb{Z} \longrightarrow Cl(X) \longrightarrow Cl(U) \to 0.$$
$$1 \longmapsto 1.Z.$$

Proof

Note that, if Y is a prime divisor in X, then, $Y \cap U$ is either empty or a prime divisor on U.

~~For~~, If $f \in K(X)^*$, and

$$(f) = \sum n_i Y_i$$

then, we can consider $f \in K(U)^*$ & write, $(f)_U = \sum n_i (Y_i \cap U)$.

... described above,

The homo, ...
is surjective, .
<u>Hint</u>. if. $D_U = \sum n_i (Y_i)_U$ is a divisor in U. then ~~condi.~~ consider
$$D = \sum n_i \overline{(Y_i)_U} .$$

(b) Left as exercise.

(c). Codim $(Z, X) = 1$ ..
Kernel$\left(Cl(X) \longrightarrow Cl(U) \right.$ consists of divisors. of divisors supported at Z.
So, the kernel is a subgroup generated by $1 \cdot Z$.

<u>Example</u>
Let Y be an irr. curve of degree d in $\mathbb{P}^2_k$.
Then, $Cl(\mathbb{P}^2 - Y) = \mathbb{Z}/d\mathbb{Z}$.

| $Y \sim dH$
H - hyperpla in $\mathbb{P}^2$.)

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