Nov 18, 2021 Lecture 16

Let k be an infinite field, $X = \mathbb{P}^n_k$, $U_0, \ldots, U_n$ the standard affine open cover of X by copies of $\mathbb{A}^n_k$, under the presentation $\mathbb{P}^n_k = \mathrm{Proj}(k[T_0, \ldots, T_n])$

Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.

Recall $\mathcal{F}(d) := \mathcal{F} \otimes \mathcal{O}(d)$

Alternately, $\mathcal{F} = \widetilde{M}$, $M = \bigoplus_{d \in \mathbb{Z}} M_d$, then $\mathcal{F}(d) = \widetilde{M(d)}$, where $(M(d))_m = M_{d+m}$.

Know: 1. $H^i(X, \mathcal{F})$ are finite dim'l k vector spaces.

2. $H^i(X, \mathcal{F}(d)) = 0$ for $i \geq 1$, $d \gg 0$.

The Euler characteristic of $\mathcal{F}$ $\chi(X, \mathcal{F}) = \chi(\mathcal{F})$ is

$$\chi(\mathcal{F}) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_k H^i(X, \mathcal{F}) = \sum_{i=0}^{n} (-1)^i h^i(\mathcal{F}).$$

where $h^i(\mathcal{F}) := \dim_k H^i(X, \mathcal{F})$.

Theorem: The function

$$m \longmapsto \chi(\mathcal{F}(m))$$

is a polynomial in m, called the Hilbert polynomial.

Proof:

This is certainly true when $n=0$, for then $X = \mathrm{Spec}\, k$ $(\mathrm{Proj}(k[T_0]) = \mathrm{Spec}\, k))$.

Assume $n > 0$, and that the assertion is true for $\mathbb{P}^{n-1}_k$.

Let $P_1, \dots, P_r \in \mathbb{P}^n_k$ be the associated points of the coherent $\mathcal{O}_X$- module $\mathcal{F}$.

Since k is infinite, we can find homogeneous linear polynomial L which does not in any of the vector spaces $\hat{P}_1, \dots, \hat{P}_r \in k[T_0, \dots, T_n]$.

Geometrically, this means we can find a hyperplane $H \hookrightarrow X = \mathbb{P}^n_k$ which does not contain the points $P_1, \dots, P_r$.

Note that

$$0 \longrightarrow \mathcal{O}_X(-1) \xrightarrow{\ L\ } \mathcal{O}_X \longrightarrow \mathcal{O}_H \longrightarrow 0$$

is exact

Since L does not contain any associated point of $\mathcal{F}$,

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}(-1) \arrow[r, "L"] & \mathcal{F} \arrow[r] & \mathcal{F}|_H \arrow[r] \arrow[d, equal] & 0 \\
& & & \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_H &
\end{tikzcd}$$

is also exact. But $H \cong \mathbb{P}^{n-1}_k$, $\mathcal{O}_X(d)|_H = \mathcal{O}_H(d)$, and $\mathcal{F}_H := \mathcal{F}|_H$ is coherent on H. By our induction hyp:

$$\chi(H, \mathcal{F}_H(m)) \text{ is a poly. in } m.$$

Note $H \overset{j}{\hookrightarrow} \mathbb{P}^n_k = X$ is an affine map, being a closed immersion. Hence $j_* \mathcal{G}$ has the same coho as $\mathcal{G}$.

Clearly $\chi(\mathcal{F}(m)) = \chi(\mathcal{F}_H(m)) + \chi(\mathcal{F}(m-1))$

This means

$$\chi(\mathcal{F}_H(m)) = \chi(\mathcal{F}(m)) - \chi(\mathcal{F}(m-1))$$

poly. in m by induction hyp.

Therefore

$$\chi(\mathcal{F}(m)) = \sum_{i=0}^{m} \chi(\mathcal{F}_H(i)) + \chi(\mathcal{F})$$

is a polynomial.

We are using the fact that $\sum_{i=0}^{m} i^d$ is polynomial in m (e.g., $\sum_{i=0}^{m} i = \frac{m(m+1)}{2}$).

Hence we are done //

Remark: This is true even if k is finite.

The trick is to make a "base change"

$$\begin{tikzcd}
v^*\mathcal{F} \quad \mathbb{P}^n_K \arrow[r, "v"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathbb{P}^n_k \quad \mathcal{F} \arrow[d] \\
\operatorname{Spec} K \arrow[r, "u"] & \operatorname{Spec} k
\end{tikzcd}$$

$k \subset K$ field ext$^{\underline{n}}$

K ~~finite~~ infinite

edit

Easy to see (use Čech cohomology)

$$H^i(\mathbb{P}^n_k, \mathcal{F}) \otimes_k K \simeq H^i(\mathbb{P}^n_K, v^*\mathcal{F})$$

Hence $\dim_K H^i(\mathbb{P}^n_K, v^*\mathcal{F}) = \dim_k H^i(\mathbb{P}^n_k, \mathcal{F})$.

Hence we've proved the theorem for all fields //

Bezout's theorem:

Let $X = \mathbb{P}^2_k = \mathrm{Proj}\left(k[T_0, T_1, T_2]\right)$

Let $C \hookrightarrow X$, $D \hookrightarrow X$ be two curves, with $\deg(C) = c$, and $\deg D = d$, i.e.

$C = Z(f)$, where f is homog of degree c

$D = Z(g)$, where g is hom of deg D.

We have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-c) \xrightarrow{f} \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

It follows that

$$\chi(\mathcal{O}_C(m)) = \chi(\mathcal{O}_X(m)) - \chi(\mathcal{O}_X(m-c))$$

Since we know coho. of $\mathbb{P}^2_k = X$,

For $m \geq 0$, $\chi(\mathcal{O}_X(m)) = h^0(\mathcal{O}_X(m))$, since $h^i(\mathcal{O}_X(m)) = 0$ $i \geq 1$.

$$= \binom{m+2}{2}$$

$$= \frac{(m+1)(m+2)}{2}$$

$$= \frac{m^2 + 3m + 2}{2}$$

Therefore

$$\mathcal{O}_C(m) = \frac{1}{2}\left\{m^2 + 3m + \cancel{2} - (m-c)^2 + 3(m-c) + \cancel{2}\right\}$$

$$= \frac{2cm + 3c - c^2}{2}$$

Next, have an exact sequence (assuming C & D share not irreducible component)

$$0 \longrightarrow \mathcal{O}_X(-d)|_C \xrightarrow{g} \mathcal{O}_C \longrightarrow \mathcal{O}_{C\cap D} \longrightarrow 0$$

$$\mathcal{O}_X(-d)|_C = \mathcal{O}_C(-d)$$

Therefore the Hilbert poly of $\mathcal{O}_{C\cap D}$ is

$$\frac{1}{2}\left\{2cm + 3c - c^2 - 2c(m-d) + 3c - c^2\right\}$$

$$= cd.$$

Example : Suppose $C \hookrightarrow \mathbb{P}^2_k = X$ is a smooth projective curve. Let Ω^1_C be the sheaf of differentials on C.

One knows by Serre duality $H^0(C, \Omega^1_C) \simeq H^1(C, \mathcal{O}_C)^*$.

The common dimension is called the genus of C, and let us denote it g.

R.R. : If L is invertible sheaf (= line bundle) on C

$$h^0(L) - h^0(\Omega^1_C \otimes L^{-1}) = \deg L + 1 - g$$

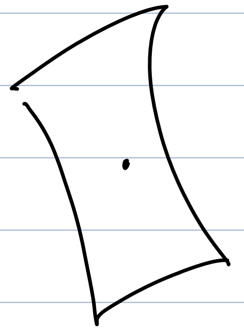
Let $L = \mathcal{O}_X(1)|_C$.

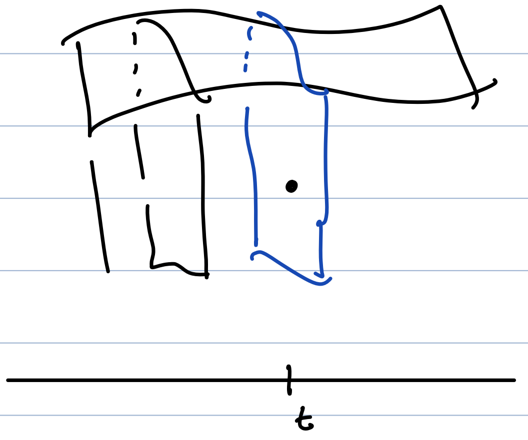
Can show :

$$g = \frac{(c-1)(c-2)}{2}$$

Using hilbert polys and above consideration.

Need $\Omega^1_C = \mathcal{O}_C(c-3)$. $\leftarrow$ adjunction formula.

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