LECTURE 15

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The symbol $\mathscr{Ab}$ will denote the category of abelian groups and $\mathbb{S}\mathrm{ch}$ the category of schemes. If X is a topological space, $\mathscr{Psh}_X$ and $\mathscr{Sh}_X$ denote the category of presheaves and the category of sheaves respectively on X. By a ring we mean a commutative ring with identity. For a ring A, Mod_A denotes the category of A-modules. For a sheaf of rings $\mathscr{A}$ on a topological space, $\mathrm{Mod}_{\mathscr{A}}$ will denote the category of $\mathscr{A}$-modules.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Noetherian Schemes

1.1. **The affine case.** The following lemma is crucial in setting up the definition of noetherian schemes.

Lemma 1.1.1. *Let A be a ring, and Γ a subset of A generating the unit ideal of A. Suppose A_f is noetherian for $f \in \Gamma$. Then A is noetherian.*

Proof. We have a finite subset $\{f_1, \dots, f_n\}$ of Γ such that $\sum_i z_i f_i = 1$ for some $z_1, \dots, z_n \in A$. It follows that $\langle f_1, \dots, f_n \rangle = A$.

Let $\varphi_i \colon A \to A_{f_i}$ be the canonical localisation map $a \mapsto a/1$. Let $\mathfrak{a}$ be an ideal in A. We claim that

$$\mathfrak{a} = \bigcap_{i=1}^{n} \varphi_i^{-1}\Big(\varphi_i(\mathfrak{a})\Big). \tag{#}$$

We only have to prove the inclusion $\bigcap_{i=1}^n \varphi_i^{-1}(\varphi_i(\mathfrak{a})) \subset \mathfrak{a}$. Let $b \in \bigcap_{i=1}^n \varphi_i^{-1}(\varphi_i(\mathfrak{a}))$. Then, for $i = 1, \dots, n$, $\varphi_i(b) = a_i/f_i^{m_i}$ for some $a_i \in \mathfrak{a}$ and some $m_i \geq 0$, whence we have $n_i \geq 0$ such that $f_i^{n_i} b \in \mathfrak{a}$. Let N be the maximum of the n_i. Then $f_i^N b \in \mathfrak{a}$ for all $i = 1, \dots, n$. Since $\langle f_1, \dots, f_n \rangle = A$, we also have $\langle f_1^N, \dots, f_n^N \rangle = A$. Thus there exist $x_1, \dots, x_n \in A$ such that $\sum_{i=1}^n x_i f_i^N = 1$. It follows that $b = b\sum_{i=1}^n x_i f_i^N = \sum_{i=1}^n x_i (f_i^N b) \in \mathfrak{a}$. This establishes (#).

Now suppose

$$\mathfrak{a}_0 \subset \mathfrak{a}_1 \subset \dots \subset \mathfrak{a}_m \subset \dots \tag{$*$}$$

is an increasing chain of ideals in A. For each $i = 1, \dots, n$, the chain $\varphi_i(\mathfrak{a}_0) \subset \varphi_i(\mathfrak{a}_1) \subset \dots \subset \varphi_i(\mathfrak{a}_m) \subset \dots$ stabilises, since A_{f_i} is noetherian. It follows that there is an $M \geq 0$ such that $\varphi_i(\mathfrak{a}_M) = \varphi_i(\mathfrak{a}_{M+m})$ for all $m \geq 0$. Applying (#) to $\mathfrak{a}_M$ and $\mathfrak{a}_{M+m}$, we see that $\mathfrak{a}_M = \mathfrak{a}_{M+m}$ for all $m \geq 0$. This proves that the chain $(*)$ stabilises, proving that A is noetherian. $\square$

An immediate corollary is

Corollary 1.1.2. *Let A be a ring and $X = \operatorname{Spec} A$.*

(a) *If A is noetherian, then the ring $\Gamma(U, \mathscr{O}_X)$ is noetherian for every affine open subscheme U of X.*
(b) *Suppose we have a cover $\mathfrak{U}$ of X by affine open subschemes such that each member of $\mathfrak{U}$ is the spectrum of a noetherian ring. Then A is a noetherian ring.*

Proof. We first prove (a). Let $f \in A$. Since $A_f = A[T]/\langle Tf - 1\rangle$, we see that A_f is noetherian by the Hilbert Basis Theorem. Let $U = \operatorname{Spec} B$ be an affine open subset of X. We can cover U by basic open sets $D_A(f)$ of X, and since $A_f = B_{f|_U}$, see that $B_{f|_U}$ is noetherian for such f. Thus $\Gamma = \{f|_U \mid f \in A \text{ and } D_A(f) \subset U\}$ generates the unit ideal of B and B_g is noetherian for all $g \in \Gamma$. By Lemma 1.1.1, B is noetherian.

Now for part (b). Let $U = \operatorname{Spec} B \in \mathfrak{U}$. Let $\psi\colon A \to B$ be the map corresponding to the open inclusion $U \subset X$. Let $f \in A$ be such that $D(f) \subset U$. By part (a), $B_{f|_U}$ is noetherian (or simply use the Hilbert Basis Theorem again), whence A_f, which equals $B_{f|_U}$, is noetherian. Now we can cover U basic open sets $D_A(f)$, and letting U vary over $\mathfrak{U}$, we can find a cover of X by affine open subschemes of the form $D_A(f)$ such that A_f is noetherian. Part (b) now follows from Lemma 1.1.1. □

1.2. **Locally noetherian and noetherian schemes.** The following result allows us to define locally noetherian schemes and noetherian in two ways in Definition 1.2.2

Proposition 1.2.1. *Let X be a scheme. The following are equivalent.*
(a) *Every affine open subscheme of X is the spectrum of a noetherian ring.*
(b) *X has a cover by affine open subschemes each of which is the spectrum of a noetherian ring.*

Proof. It is clear that (a) ⇒ (b). We now prove (b) ⇒ (a). Let $\mathfrak{V}$ be a cover of X as in the hypothesis of (b). Let U be an affine open subscheme of X. Then according to Corollary 1.1.2 (a), affine open subschemes of $U \cap V$ are spectrums of noetherian rings for every $V \in \mathfrak{V}$. It follows, by varying V in $\mathfrak{V}$, that U can be covered by affine open subschemes of noetherian schemes. By Corollary 1.1.2 (b), we see that U is the spectrum of a noetherian scheme. □

Definition 1.2.2. A scheme X is said to be *locally noetherian* if it satisfies any of the two equivalent conditions in Proposition 1.2.1. It is said to be *noetherian* if it is locally noetherian and quasi-compact.

The following result is obvious given the earlier results and the definition of a noetherian scheme.

Proposition 1.2.3. *A scheme X is noetherian if and only if X has a finite open cover by affine open subschemes which are the spectrums of noetherian rings.*

Proof. Obvious. □

Example 1.2.4. Let A be a ring and $X = \mathbb{P}^n_A$. Writing $A[\mathbf{T}]$ as a short hand for $A[T_0, \dots, T_n]$ and representing X as $\operatorname{Proj}(A[T_0, \dots, T_n])$, we see by the Hilbert Basis Theorem that for $i \in \{1, \dots, n\}$, $U_i = \operatorname{Spec}(A[\mathbf{T}]_{(T_i)})$ is the spectrum of a noetherian ring, and since the U_i, $i = 0, \dots, n$, cover X, X is noetherian. Conversely, suppose X is noetherian. Then U_0 being an affine open subscheme, is the spectrum of a noetherian ring, namely $A[\mathbf{T}]_{(T_0)} = A[\mathbf{T}/T_0] \cong A[Y_1, \dots, Y_n]$. It follows that $A \cong A[Y_1, \dots, Y_n]/\langle Y_1, \dots, Y_n\rangle$ is noetherian. Thus $\mathbb{P}^n_A$ *is noetherian if and only of* A *is noetherian*. In fact, a little thought shows that $\mathbb{P}^n_A$ *is locally noetherian if and only if* A *is noetherian*, since $\mathbb{P}^n_A$ is quasi-compact, being the finite union of the quasi-compact open subschemes U_i, $i = 1, \dots, n$.

2. Coherent sheaves

2.1. **Definitions.** We give the following general definitions (see Section 01BU of the Stacks Project) for the sake of completeness, though we will only use these definitions for schemes, and for coherent sheaves, only when the underlying space is locally noetherian, where the definition simplifies considerably. Let X be a ringed space. A sheaf $\mathscr{F}$ on X is said to be of *finite type* on X if $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_X}$ and for every $x \in X$ there exists an open neighbourhoood U of x and an epimorphism $\mathscr{O}_U^n \twoheadrightarrow \mathscr{F}|_U$ of $\mathscr{O}_U$-modules. An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be *coherent* if

- **(a)** $\mathscr{F}$ is of finite type, and
- **(b)** for every open $U \subset X$ and every map $\mathscr{O}_U^n \to \mathscr{F}|_U$, the kernel is of finite type as an $\mathscr{O}_U$-module. Note that it is not required that the local maps $\mathscr{O}_U^n \to \mathscr{F}|_U$ be surjective.

Note that the definition of coherent is different from that given in [H].

When X is a locally noetherian scheme, an $\mathscr{O}_X$-module $\mathscr{F}$ is coherent if and only if it is quasi-coherent and $\Gamma(U, \mathscr{F})$ is a finitely generated $\Gamma(U, \mathscr{O}_X)$-module for every affine open U of X. See Section 01XY of [SP], especially, Lemma 30.9.1 for details. In this case the definition we have given agrees with the one in [H].

In view of the above, for this course, we give the following definition

Definition 2.1.1. Let X be a locally noetherian scheme. An $\mathscr{O}_X$-module $\mathscr{F}$ is said to be *coherent* if it is quasi-coherent and on each affine open subscheme U, $\mathscr{F}|_U = \widetilde{M}$ for some finitely generated $\Gamma(U, \mathscr{O}_X)$-module M.

2.1.2. Convention. If we say $\mathscr{F}$ is coherent on a scheme X, it will be assumed that X is locally noetherian. In particular, if X is known to be quasi-compact, for example $X = \mathbb{P}^n_A$, then, by our convention, if X has a coherent sheaf on it, the underlying assumption is that X is noetherian.

Theorem 2.1.3. *Let X be a locally noetherian schemes and $\mathscr{F}$ a quasi-coherent sheaf on X. Then $\mathscr{F}$ is coherent if and only if there exists an affine open cover $\mathfrak{U}$ of X such that $\Gamma(U, \mathscr{F})$ is a $\Gamma(U, \mathscr{O}_X)$-module of finite type for every $U \in \mathfrak{U}$.*

Proof. We omit the proof since it is very similar to the proof of Proposition 1.2.1. We however make two observations.

- **(a)** $\Gamma(U, \mathscr{O}_U)$ is a noetherian ring for every affine open subscheme U.
- **(b)** If A is a noetherian ring, M an A-module, $f_1, \dots, f_d$ a finite set of elements of A such that $\langle f_1, \dots, f_d\rangle = A$ and such that M_{f_i} is a noetherian A_{f_i}-module for $i = 1, \dots, d$, then M is a noetherian A-module. Indeed, we may choose a finite set of generators of the A_{f_i}-module M_{f_i} to be the images of elements from M. Then we have a finite number of elements $m_{ij} \in M$ such for each for each i, the images of m_{ij} in M_{f_i} as j varies, forms a set of generators for M_{f_i} as an A_{f_i}-module. Then the m_{ij} form generators of of the A-module M. Indeed, the finite rank free module $\bigoplus_{i,j} A$ maps to M via $(x_{ij}) \mapsto \sum_{i,j} x_{ij} m_{ij}$, and this map is surjective.

These two observations together with the proof of Proposition 1.2.1 should help in completing the proof. ∎

3. Direct image of coherent sheaves

3.1. **Localisation results.** The following basic result should be regarded as a generalisation of statements made in the proof of Proposition 1.4.2 of Lecture 12, especially the assertion that $\Gamma(X,\, \mathscr{O}_X)_g \cong \Gamma(X_g,\, \mathscr{O}_{X_g})$ for X quasi-compact and $g \in \Gamma(X,\, \mathscr{O}_X)$ (this is the assertion that the map θ_g of *loc.cit.* is an isomorphism).

Theorem 3.1.1. *Let X be a scheme, $\mathscr{L}$ and $\mathscr{F}$ quasi-coherent $\mathscr{O}_X$-modules with $\mathscr{L}$ invertible, and f a section of $\mathscr{L}$ over X. Let $X_f = \{x \in X \mid f_x \notin \mathfrak{m}_x\mathscr{L}\}$.*

(a) *Let X be quasi-compact. Suppose $s \in \Gamma(X,\, \mathscr{F})$ is such that $s|_{X_f} = 0$. Then there exists $n \geq 0$ such that $f^n s = 0$, where $f^n s$ is regarded as a section of $\mathscr{L}^{\otimes n} \otimes \mathscr{F}$ over X.*

(b) *Suppose X can be covered by a finite number of quasi-compact open subschemes $U_1, \ldots, U_d$ such that for each i we have $\mathscr{L}|_{U_i} \cong \mathscr{O}_{U_i}$, and for i and j, $U_i \cap U_j$ is quasi-compact.*[1] *Let $s \in \Gamma(X_f,\, \mathscr{F})$. Then there exists $n \geq 0$ such that $f^n s$, regarded as a section of $\mathscr{L}^{\otimes n} \otimes \mathscr{F}$ over X_f, extends to a section of $\mathscr{L}^{\otimes n} \otimes \mathscr{F}$ over X.*

Proof. Let us prove (a). We can cover X by a finite number of quasi-compact open subschemes V_j, $j = 1, \ldots, e$, such that for each j, $\mathscr{L}|_{V_j} \cong \mathscr{O}_{V_j}$. Fix $V \in \{V_1, \ldots, V_e\}$, Let $g = f|_V$. We may regard g as a section of $\mathscr{O}_V$. Observe that $V_g = X_f \cap V$. Since V is quasi-compact, from the isomorphism (1.4.4.1) of Lecture 12, we see that $\Gamma(V_g, \mathscr{F}|_{V_g}) = \Gamma(V,\, \mathscr{F})_g$. Since the image of s in $\Gamma(V_g, \mathscr{F}|_{V_g})$ is zero, we see that for some $m = m_U$, we have $g^m(s|_U) = 0$. Taking n to be the maximum of the m_U as U varies in $\{V_j\}_{j=1}^e$, we see that $f^n s = 0$.

Part (b) is similar. Let $U \in \{U_1, \ldots, U_d\}$. Let $g = f|_U$. Since $\mathscr{L}|_U \cong \mathscr{O}_U$, we may regard g as a section of $\mathscr{O}_U$. Once again, by *loc.cit.* we have $\Gamma(U,\, \mathscr{F})_g \xrightarrow{\sim} \Gamma(U_g,\, \mathscr{F})$. Since $X_f \cap U = U_g$, therefore for some $m = m_U \geq 0$, $g^m(s|_{U_g})$ extends to a section of $\mathscr{O}_U$ over U. Let m be the maximum of the m_U as U ranges over $\{U_i\}_{i=1}^d$. In other words for $i = 1, \ldots, d$, $f^m(s|_{X_f \cap U_i})$ extends to a section of $\mathscr{L}^{\otimes m} \otimes \mathscr{F}$. Let $W_i = \bigcup_{k=1}^i U_k$, $i = 1, \ldots, d$. Let $i \in \{1, \ldots, d\}$. Suppose we have $n_i \geq 0$ such that $f^{n_i}(s|_{W_i \cap X_f})$ extends to a section of $\mathscr{L}^{\otimes n_i} \otimes \mathscr{F}|_{W_i}$. Note that for $i = 1$, $n_1 = m$ satisfies the requirement. If $i = d$, we are done, since $W_d = X$. So suppose $1 \leq i < d$. By replacing n_i by a larger integer if necssary, we may assume that $n_i \geq m$. Let σ be an extension of $f^{n_i}(s|_{W_i \cap X_f})$ to W_i and τ an extension of $f^{n_i}(s|_{U_{i+1} \cap X_f})$ to U_{i+1}. Now σ and τ agree on $X_f \cap W_i \cap U_{i+1}$. Now $W_i \cap U_{i+1} = \bigcup_{k=1}^i (U_k \cap U_{i+1})$, whence $W_i \cap U_{i+1}$ is quasi-compact by our hypothesis on the U_j. Hence applying part (a) to $\sigma|_{W_i \cap U_{i+1} \cap X_f} - \tau|_{W_i \cap U_{i+1} \cap X_f}$, we see that there is a non-negative integer n_{i+1} such that $f^{n_{i+1}}(s|_{W_{i+1} \cap X_f})$ extends to W_{i+1}. By induction, we are done, since $W_d = X$. ∎

3.2. **More coherent sheaves.** We begin with an important definition.

[1] This happens for example if X is separated and quasi-compact, for then X can be covered by a finite number of affine opens, whose pair-wise intersection is also affine. More generally, this is true when X is quasi-compact, and its diagonal immesrion is a quasi-compact map.

Definition 3.2.1. Let X be a scheme and $\mathscr{F}$ a quasi-coherent $\mathscr{O}_X$-module. We say $\mathscr{F}$ *is generated by global sections* if there exist sections $s_i \in \Gamma(X, \mathscr{F})$, $i \in I$, such that the $\mathscr{O}_{X,x}$-module $\mathscr{F}_x$ is generated by the germs $\{s_{i,x}\}_{i\in I}$. Equivalently, we have an epimorphism of $\mathscr{O}_X$-modules $\bigoplus_{i\in I} \mathscr{O}_X \twoheadrightarrow \mathscr{F}$. If the index set I is finite, then we say $\mathscr{F}$ is *generated by a finite number of global sections.*

Recall from Example 1.2.4 that if A is a ring, then $\mathbb{P}^d_A$ is locally noetherian $\Leftrightarrow$ $\mathbb{P}^n$ is neotherian $\Leftrightarrow$ A is noetherian. Therefore it makes sense to talk about coherent sheaves on $\mathbb{P}^d_A$ only when A is noetherian. The following theorem due to Serre is one of the corner stones of sheaf-theoretic algebraic geometry

Theorem 3.2.2. (Serre) *Let A be a noetherian ring, and $X = \mathbb{P}^d_A$, $d \geq 0$. Let $\mathscr{F}$ be a coherent sheaf on X.*

(a) *There exists $n_\circ = n_\circ(\mathscr{F}) \geq 0$ such that for $n \geq n_\circ$, $\mathscr{F}(n)$ is generated by a finite number of global sections.*
(b) *For all i, $\mathrm{H}^i(X, \mathscr{F})$ is an A-module of finite type (i.e. it is finitely generated).*
(c) *There exists $n_\circ = n_\circ(\mathscr{F}) \geq 0$ such that for all $n \geq n_\circ$, and for all $i \geq 1$, $\mathrm{H}^i(X, \mathscr{F}(n)) = 0$.*

Proof. We first prove (a). Let $T_0, \ldots, T_d$ be algebraically independent variables over A. Write $A[\mathbf{T}] = A[T_0, \ldots, T_d]$ and $X = \mathrm{Proj}(A[\mathbf{T}])$. For $i \in \{0, \ldots, d\}$ let $U_i = D_+(T_i) = \mathrm{Spec}(A[\mathbf{T}]_{(T_i)} = \mathrm{Spec}(A[\mathbf{T}/T_i])$, where the notation is obvious. Set $M_i = \Gamma(U_i, \mathscr{F})$. Since M_i is a finitely generated $B_i := A[\mathbf{T}/Ti]$-module, we have a finite set of generators $\{m_{ij}\}$ for the B_i-module M_i. Let $\mathscr{L} = \mathscr{O}_X(1)$ and regard T_i as a global section of $\Gamma(X, \mathscr{L})$. Using the notations of Theorem 3.1.1, $D_+(T_i)$ equals X_{T_i}. Hence, by Theorem 3.1.1, there exist $n_{ij} \geq 0$ such that $T_i^{n_{ij}} m_{ij}$ extends to a global section of $\mathscr{L}^{\otimes n_{ij}} \otimes \mathscr{F}$. By taking $n_\circ$ to be the maximum of n_{ij}, which are finite in number, we see that $T_i^{n_\circ} m_{ij}$ extends to a global section $t_{ij} \in \Gamma(X, \mathscr{L}^{\otimes n_\circ} \otimes \mathscr{F})$. Since, by definition $\mathscr{L}^{\otimes n_\circ} \otimes \mathscr{F} = \mathscr{F}(n_\circ)$, we have found a finite number of global sections t_{ij} of $\Gamma(X, \mathscr{F}(n_\circ))$. It is clear that these global sections generate $\mathscr{F}(n_\circ)$. Moreover, on examining the proof, it is clear that $\mathscr{F}(n)$ is generated by global sections for $n \geq n_\circ$.

For parts (b) and (c) note that by (a) we have $q \geq 0$ and an epimorphism of $\mathscr{O}_X$-modules $\bigoplus_{i=1}^d \mathscr{O}_X \twoheadrightarrow \mathscr{F}(q)$. Setting $\mathscr{E} = \bigoplus_{i=1}^d \mathscr{O}_X(-q)$, we get an exact sequence

$$(*) \qquad 0 \longrightarrow \mathscr{R} \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow 0$$

Note that $\mathscr{R}$ is also coherent, since $\mathscr{E}$ and $\mathscr{F}$ are.

We now prove (b). From the theorem on cohomology of projective space, the assertion is true for $\mathscr{E}$. Since X is separated and quasi-finite, $\mathrm{H}^i(X, \mathscr{G}) = \check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{G})$ for any quasi-coherent sheaf $\mathscr{G}$ with $\mathfrak{U}$ the ordered affine open cover given by $U_0, \ldots, U_d$. It follows that $\mathrm{H}^i(X, \mathscr{G}) = 0$ for $i \geq d+1$ for such $\mathscr{G}$. Now suppose we have $j \geq 0$ such that for every coherent $\mathscr{G}$, $\mathrm{H}^i(X, \mathscr{G})$ is finitely generated for $i \geq j+1$. From $(*)$ we have the exact sequence

$$(\#) \qquad \ldots \longrightarrow \mathrm{H}^j(X, \mathscr{E}) \longrightarrow \mathrm{H}^j(X, \mathscr{F}) \longrightarrow \mathrm{H}^{j+1}(X, \mathscr{R}) \longrightarrow \ldots.$$

Since $\mathrm{H}^j(X, \mathscr{E})$ and $\mathrm{H}^{j+1}(X, \mathscr{R})$ are finitely generated A-modules, $\mathrm{H}^j(X, \mathscr{F})$ is also finitely generated. By descending induction, we are done.

For (c) we use the exact sequence

$$(**) \qquad 0 \longrightarrow \mathscr{R}(n) \longrightarrow \mathscr{E}(n) \longrightarrow \mathscr{F}(n) \longrightarrow 0$$

for $n \in \mathbf{Z}$ and the resulting exact sequence

$$(\#\#) \qquad \ldots \longrightarrow \mathrm{H}^j(X, \mathscr{E}(n)) \longrightarrow \mathrm{H}^j(X, \mathscr{F}(n)) \longrightarrow \mathrm{H}^{j+1}(X, \mathscr{R}(n)) \longrightarrow \ldots .$$

Now for $n \geq q+1$, we know from the theorem on cohomology of projective space that $\mathrm{H}^i(X, \mathscr{E}(n)) = 0$ for $i \geq 1$.

Suppose we have $j \geq 1$ such that for every coherent $\mathscr{G}$ on X there exists $N = N_{j+1}(\mathscr{G})$ such that $\mathrm{H}^i(X, \mathscr{G}(n)) = 0$ for $i \geq j+1$ and $n \geq N$. (We have just observed that this condition is true for $j = d$.) From the exact sequence (##) we see that for $n \geq N_{j+1}(\mathscr{R})$, the relation $\mathrm{H}^j(X, \mathscr{F}(n)) = 0$ holds. By descending induction, for each $j \geq 1$ we have $N_j(\mathscr{F})$ such that $\mathrm{H}^i(X, \mathscr{F}(n)) = 0$ for $i \geq j$ and $n \geq N_j(\mathscr{F})$. Since we only have to worry about $j = 1, \ldots, d$, (c) follows. □

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