LECTURE 14

Date of Lecture: Nov 11, 2021

The symbol $\mathcal{Ab}$ will denote the category of abelian groups and $\mathbb{S}$ch the category of schemes. If X is a topological space, $\mathcal{Psh}_X$ and $\mathcal{Sh}_X$ denote the category of presheaves and the category of sheaves respectively on X. By a ring we mean a commutative ring with identity. For a ring A, Mod_A denotes the category of A-modules. For a sheaf of rings $\mathscr{A}$ on a topological space, $\mathrm{Mod}_{\mathscr{A}}$ will denote the category of $\mathscr{A}$-modules.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

Conventions for this lecture. Throughout this lecture A will be a ring, *not* necessarily noetherian, n a positive integer, $R = A[T_0, \ldots, T_n]$, the polynomial ring in $(n+1)$-variables over A, regarded as a graded ring with the usual grading $R = \bigoplus_{d\geq 0} R_d$, with $R_0 = A$, and R_d the free A-module of homogeneous degree d polynomials in $T_0, \ldots, T_n$ with coefficients in A. We use the standard symbols $\mathbb{A}^{n+1}_A$ and $\mathbb{P}^n_A$ for $\mathrm{Spec}\, R$ and $\mathrm{Proj}(R)$ respectively. The scheme $\mathbb{A}^{n+1}$ is called *the relative affine space over A of relative dimension* $n+1$ and $\mathbb{P}^n_A$ *the relative projective space over A of relative dimension n.*

1. The Punctured Affine Space over A

1.1. **Notations.** For $\boldsymbol{\mu} = (\mu_0, \ldots, \mu_n) \in \mathbf{Z}^{n+1}$, we use the following notations and conventions

1. $|\boldsymbol{\mu}| := \sum_{i=1}^{n} \mu_i$.
2. $\mathbf{T}^{\boldsymbol{\mu}} := T_0^{\mu_0} \ldots T_n^{\mu_n}$. (The negative powers occur in localisations of R.)
3. If $\boldsymbol{\nu} = (\nu_0, \ldots, \nu_n)$ is another element of $\mathbf{Z}^{n+1}$, then the notation $\boldsymbol{\mu} \leq \boldsymbol{\nu}$ (respectively $\boldsymbol{\mu} < \boldsymbol{\nu}$) if $\mu_i \leq \nu_i$ (respectively $\mu_i < \nu_i$) for $i = 0, \ldots, n$. Thus
$$R_d = \bigoplus_{\boldsymbol{\mu}\geq\mathbf{0},|\boldsymbol{\mu}|=d} A \cdot \mathbf{T}^{\boldsymbol{\mu}}, \quad d \geq 0.$$
4. As always, $R_+ = \bigoplus_{d>0} R_d$. Note that $R_+ = \langle T_0, \ldots, T_d \rangle$, the ideal in R generated by the variables $T_0, \ldots, T_d$.
5. P will denote the R-module of *inverse polynomials* defined in Problem 5 of Homework 3. Recall that as an A-module,
$$P = \bigoplus_{\boldsymbol{\nu}<\mathbf{0}} A \cdot \mathbf{T}^{\boldsymbol{\nu}}.$$
Its R-module structure is given as follows: For $\boldsymbol{\mu} \geq \mathbf{0}$ and $\boldsymbol{\nu} < \mathbf{0}$ (so that $\mathbf{T}^{\boldsymbol{\mu}} \in R$ and $\mathbf{T}^{\boldsymbol{\nu}} \in P$)

$$\mathbf{T}^{\boldsymbol{\mu}} \cdot \mathbf{T}^{\boldsymbol{\nu}} = \begin{cases} \mathbf{T}^{\boldsymbol{\mu}+\boldsymbol{\nu}} & \text{if } \boldsymbol{\mu}+\boldsymbol{\nu} < \mathbf{0} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1.1}$$

6. In fact (1.1.1) shows that P is a graded R-module. In greater detail let

$$P_d = \bigoplus_{\boldsymbol{\nu}<\mathbf{0},\, |\boldsymbol{\nu}|=d} A\cdot \mathbf{T}^{\boldsymbol{\nu}}. \tag{1.1.2}$$

Then $P = \bigoplus_d P_d$, and it is clear from (1.1.1) that with this grading, P is a graded R-module.

1.1.3. Note that $P_d = 0$ for $d > -n-1$. Indeed, if $\boldsymbol{\nu} < \mathbf{0}$, then $|\boldsymbol{\nu}| \le -n-1$. On the other hand if $d \le -n-1$ then $P_d \neq 0$.

1.2. **The cohomology of the punctured affine space.** Let $Z = \operatorname{Spec} R/R_+ = V(R_+)$. If A is a field k, then Z would represent the origin in the affine space $\mathbb{A}_k^{n+1}$. In the more general situation that we are discussing, it is regarded Z is regarded a relative origin, and Z is a copy of $\operatorname{Spec} A$ in $\mathbb{A}_A^{n+1}$, since $R/\langle \mathbf{T}\rangle = A$. Another way of thinking about this is that we have a natural map $\pi\colon \mathbb{A}_A^{n+1} \to \operatorname{Spec} A$ and a section $\operatorname{Spec} A \hookrightarrow \mathbb{A}_A^{n+1}$ of π whose image is Z. Our interest is in the scheme

$$V := \mathbb{A}_A^{n+1} \smallsetminus Z. \tag{1.2.1}$$

For $i = 0, \ldots, n$ set

$$V_i := D(T_i) = \operatorname{Spec}(R_{T_i}). \tag{1.2.2}$$

Then $\mathfrak{V} = \{V_i\}_{i=0}^n$ is an affine open cover of V. Now $\mathbb{A}_A^{n+1}$ is separated, being an affine scheme, and hence so is V, being an open subscheme of an affine scheme. Therefore, according to Theorem 3.2.4 of Lecture 9, we have natural isomorphisms

$$\check{\mathrm{H}}^i(\mathfrak{V},\, \mathscr{O}_V) \xrightarrow{\sim} \mathrm{H}^i(V,\, \mathscr{O}_V), \quad i \ge 0. \tag{1.2.3}$$

Now the module of Čech p-cochains $C^p(\mathfrak{V},\, \mathscr{O}_V)$ is the direct sum of localisations of R by homogeneous elements and hence is graded. More precisely

$$C^p(\mathfrak{V},\, \mathscr{O}_V) = \bigoplus_{i_0<\cdots<i_p} R_{T_{i_o}\ldots T_{i_p}} \tag{1.2.4}$$

and since the T_i are homogeneous elements, (1.2.4) shows that $C^p(\mathfrak{V},\, \mathscr{O}_V)$ is graded. Let the representation of $C^p(\mathfrak{V},\, \mathscr{O}_V)$ as a graded R-module be

$$C^p(\mathfrak{V},\, \mathscr{O}_V) = \bigoplus_{d\in\mathbf{Z}} C^p_d. \tag{1.2.5}$$

From the definition of the coboundary maps in $C^\bullet(\mathfrak{V},\, \mathscr{O}_V)$, it is clear that these coboundary maps respect the grading in (1.2.5). It follows that $\check{\mathrm{H}}^p(\mathfrak{V},\, \mathscr{O}_V)$ is a graded R-module. Let the decomposition of $\check{\mathrm{H}}^p(\mathfrak{V}, \mathscr{O}_V)$ as a graded R-module be written as follows:

$$\check{\mathrm{H}}^p(\mathfrak{V}, \mathscr{O}_V) = \bigoplus_{d\in\mathbf{Z}} \mathrm{H}^p_d. \tag{1.2.6}$$

According to Problem 5 of Homework 3

$$\mathrm{H}^i_d = \begin{cases} R_d & \text{if } i = 0 \\ P_d & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.7}$$

Now the graded R-module structure of P gives us

$$R_d \cdot P_{-n-1-d} \subset P_{-n-1} = A\cdot T_0^{-1}\ldots T_n^{-1} \xrightarrow{\sim} A$$

for every $d \in \mathbf{Z}$. For $d \in \mathbf{Z}$ let $d^* = -n-1-d$. The above observations give us a pairing (via scalar multiplication) of free A-modules

$$R_d \times P_{d^*} \longrightarrow P_{-n-1} = A \cdot T_0^{-1} \dots T_n^{-1} \xrightarrow{\sim} A. \tag{1.2.8}$$

This pairing is perfect, as we now prove. When $d < 0$, then $d^* = -n-1-d > -n-1$, whence both R_d and P_{d^*} are zero (see Remark in 1.1.3), and hence the pairing is perfect in this case. Now assume $d \geq 0$, so that $d^* \leq -n-1$. For $\boldsymbol{\nu} = (\nu_0, \dots, \nu_n)$ let $\boldsymbol{\nu}^* = (\nu_0^*, \dots, \nu_n^*)$ be the $(n+1)$-tuple defined by the relation $\nu_i^* = -1 - \nu_i$. Let $\mathbf{T}^{\boldsymbol{\mu}}$ be a basis element of R_d under the standard representation of R_d as a free A-module, and $\mathbf{T}^{\boldsymbol{\nu}}$ a basis element of P_{d^*} under the standard representation of P_{d^*} as free A-module. This means $\boldsymbol{\mu} \geq \mathbf{0}$ and $|\boldsymbol{\mu}| = d$, and $\boldsymbol{\nu} < \mathbf{0}$ and $|\boldsymbol{\nu}| = d^*$. It is easy to see $\boldsymbol{\mu} + \boldsymbol{\nu} < \mathbf{0}$ if and only if $\boldsymbol{\nu}^* = \boldsymbol{\mu}$. From the formula given in (1.1.1) it follows that

$$\mathbf{T}^{\boldsymbol{\mu}} \cdot \mathbf{T}^{\boldsymbol{\nu}} = \delta_{\boldsymbol{\mu}, \boldsymbol{\nu}^*} T_0^{-1} \dots T_n^{-1} \tag{1.2.9}$$

where $\delta_{\boldsymbol{\mu}, \boldsymbol{\nu}^*}$ is the Kronecker symbol. It follows that (1.2.8) is a perfect pairing of free A-modules.

One consequence of the above discussion is the following:

Proposition 1.2.10. *Let $n \geq 1$ and V the open subscheme of $\mathbb{A}_A^{n+1}$ defined above. Then V is not affine*

Proof. We have $\mathrm{H}^n(V, \mathscr{O}_V) \cong P \neq 0$. Since $n \geq 1$, by Serre's Theorem (see Theorem 1.3.1 of Lecture 12) we are done. □

1.2.11. However, the punctured relative affine line $\mathbb{A}_A^1 \smallsetminus Z$ is affine . In greater detail, writing $\mathbb{A}_A^1$ as $\operatorname{Spec} A[T]$, and $Z = V(\langle T \rangle)$, the scheme $\mathbb{A}_A^1 \smallsetminus Z$ is affine. In fact it is equal to the spectrum of the localisation $A[T]_T$.

2. Relative projective space

2.1. **An affine open cover for $\mathbb{P}_A^n$.** We retain the above notations. For notational ease we write $X = \mathbb{P}_A^n$ for the rest of this lecture. For $i = 0, \dots, n$ set

$$U_i = D_+(T_i). \tag{2.1.1}$$

It is clear that $\mathfrak{U} = \{U_i\}_{i=0}^n$ is an open cover of $\mathbb{P}_A^n$, since the T_i generate R_+.

Since $D_+(T_i)$ is the spectrum of $R_{(T_i)}$, where $R_{(T_i)}$ is the ring of homogeneous degree zero elements in the graded ring R_{T_i}, it is easy to identify U_i with $\mathbb{A}_A^n$. In greater detail, suppose $\xi = p(\mathbf{T})/T_i^m \in R_{T_i}$ is homogeneous, with $p(\mathbf{T}) \in R = A[\mathbf{T}]$. Then $p(\mathbf{T})$ is necessarily homogeneous, and $\deg \xi = \deg p(\mathbf{T}) - m$. In particular $\xi \in R_{(T_i)}$ if and only if $\deg p(\mathbf{T}) = m$. Since $p(\mathbf{T})$ is homogeneous of degree m, $p(\lambda\mathbf{T}) = \lambda^m p(\mathbf{T})$ where λ is in any A-algebra containing $R = A[\mathbf{T}]$. In particular $\xi = p(\mathbf{T}/T_i)$. Thus

$$A[\mathbf{T}]_{(T_i)} = A[T_0/T_i, \dots, T_n/T_i] = A[T_0/T_i, \dots, \widehat{T_i/T_i}, \dots, T_n/T_i],$$

where the "hat" over the i^{th} entry means, as usual, that that entry is omitted. It is easy to see that for $j \neq i$, the elements T_j/T_i are algebraically independent over A. Thus

$$A[\mathbf{T}]_{(T_i)} \xrightarrow{\sim} A[y_1, \dots, y_n] \tag{2.1.2}$$

where the y_j are algebraically independent. In particular

$$U_i = D_+(T_i) \xrightarrow{\sim} \operatorname{Spec}(A[y_1, \dots, y_n]) = \mathbb{A}_A^n. \tag{2.1.3}$$

2.1.4. At this point, if you wish to shore up your intuition look at Figures 1–5 in Lecture 6. The horizontal brown plane in Figures 1–3 is $D_+(T_2)$ and vertical green plane in Figures 3 and 4 is $D_+(T_0)$. Here $x = T_0$, $y = T_1$ and $z = T_2$. The ring A is $\mathbf{R}$ in the pictures. In the general case, we regard $\mathbb{A}^n_A$ as a family $\{\mathbb{A}^n_{\kappa(\mathfrak{p})}\}$ of affine n-spaces indexed by points $\mathfrak{p}$ in $\operatorname{Spec} A$, with $\kappa(\mathfrak{p})$ the residue field of A at $\mathfrak{p}$.

2.2. **The sheaves $\mathscr{O}_X(d)$.** For $d \in \mathbf{Z}$ set $R(d)$ equal to the graded R-module whose m^{th} graded piece is the free A-module R_{d+m}, the space of homgeneous polynomials over A in $\mathbf{T}$ of degree $d + m$. The quasi-coherent $\mathscr{O}_X$-module $\mathscr{O}_X(d)$ is defined by the formula

$$\mathscr{O}_X(d) = \widetilde{R(d)}. \tag{2.2.1}$$

Now $\Gamma((D_+(T_i),\, \mathscr{O}_X(d)) = (R(d))_{(T_i)}$, and hence a typical section of $\mathscr{O}_X(d)$ over $D_+(T_i)$ is a fraction of the form $\xi = p(\mathbf{T})/T_i^m$ where $p(\mathbf{T}) \in R$ is homogeneous of degree $m + d$. Therefore $\xi = T_i^d p(\mathbf{T})/T_i^{m+d} = T_i^d p(\mathbf{T}/T_i) \in T_i^d A[\mathbf{T}/T_i] \xrightarrow{\sim} A[\mathbf{T}/T_i]$ Since $A[\mathbf{T}/T_i] = R_{(T_i)} = \Gamma(D_+(T_i),\, \mathscr{O}_X)$, it follows that we have an isomorphism

$$\mathscr{O}_X(d)|_{D_+(T_i)} \xrightarrow{\sim} \mathscr{O}_X|_{D_+(T_i)} \tag{2.2.2}$$

Thus $\mathscr{O}_X(d)$ looks locally like the structure sheaf, but may not look so globally. Such sheaves are called *invertible sheaves*. We will say more about them later.

Let $0 \le i_0 < \cdots < i_p \le n$ and for such a choice of indices set

$$U_{i_0 \dots i_p} = U_{i_0} \cap \cdots \cap V_{i_p}. \tag{2.2.3}$$

By definition of $\mathscr{O}_X(d)$, we have

$$\Gamma(U_{i_0\dots i_p},\, \mathscr{O}_X(d)) = R(d)_{(T_{i_0}\cdots T_{i_p})} = \Big(R(d)_{T_{i_0}\cdots T_{i_p}}\Big)_0,$$

where the right most module is the degree zero component of the graded module $R(d)_{T_{i_0}\cdots T_{i_p}}$. In other words,

$$\Gamma(U_{i_0\dots i_p},\, \mathscr{O}_X(d)) = \Gamma(V_{i_0\dots i_p},\, \mathscr{O}_V)_d \tag{2.2.4}$$

where $V_{i_0\dots i_p} = V_{i_0} \cap \ldots \cap V_{i_p}$, and, as usual, the subscript d on the right indicates the d^{th} graded piece of a graded module. Here V and V_i are the schemes in § 1.2.

2.3. **Cohomology of projective space.** By Theorem 2.1.6 of Lecture 13 we see that $X = \mathbb{P}^n_A$ is separated. Thus $\mathfrak{U} = \{U_i\}_{i=0}^n$ is an affine open cover of the separated scheme X, and hence according to Theorem 3.2.4 of Lecture 13 we have a canonical isomorphism

$$\check{\mathrm{H}}^i(\mathfrak{U}\mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i(X,\, \mathscr{F}) \tag{2.3.1}$$

for each $i \ge 0$. Using (2.2.4) and the notation for the direct sum decomposition of the p^{th} piece of the Čech complex $C^\bullet(\mathfrak{V},\, \mathscr{O}_V)$ in (1.2.5), we get

$$C^p(\mathfrak{U},\, \mathscr{O}_X(d)) = C^p_d \tag{2.3.2}$$

for every $d \in \mathbf{Z}$. Recall that the coboundaries ∂^p_V in $C^\bullet(\mathfrak{V},\, \mathscr{O}_V)$ respect the various decompositions $C^p(\mathfrak{V},\, \mathscr{O}_V) = \bigoplus_d C^p_d$ in (1.2.5). Let $\partial^p_{V,\,d}$ denote the d^{th} piece of ∂^p_V. Then it is easy to see that $\partial^p_{V,\,d}$ is the p^{th} coboundary of $C^\bullet(\mathfrak{U},\, \mathscr{O}(d))$ under the canonical identification in (2.3.2). It follows that

$$\check{\mathrm{H}}^i(\mathfrak{U},\, \mathscr{O}_X(d)) = \mathrm{H}^i_d \tag{2.3.3}$$

where H^i_d is as in (1.2.6). From (1.2.7) we conclude that

$$\check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X(d)) = \begin{cases} R_d & \text{if } i = 0 \\ P_d & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases} \tag{2.3.4}$$

Recall that $R_d = 0$ for $d < 0$ and $P_d = 0$ for $d > -n-1$. Moreover, we have a perfect pairing of free A-modules $R_d \times P_{d^*} \to A$ where $d^* = -n-1-d$ (see (1.2.8) and the argument below that). This gives us the following theorem, which is the result in [H, Theorem 5.1, p.225]. However we do not assume in our proof that A is noetherian.

Theorem 2.3.5. (Cohomology of projective space) *With the above notations we have*

(a) *The natural map* $R \to \bigoplus_{d\in\mathbf{Z}} \mathrm{H}^0(X, \mathscr{O}_X(d))$ *is an isomorphism of graded R-modules.*
(b) $\mathrm{H}^i(X, \mathscr{O}_X(d)) = 0$ *for* $0 < i < n$ *and all* $d \in \mathbf{Z}$.
(c) $\mathrm{H}^n(X, \mathscr{O}_X(-n-1)) \xrightarrow{\sim} A$.
(d) *For each* $d \in \mathbf{Z}$, *there is a natural pairing*

$$\mathrm{H}^0(X, \mathscr{O}_X(d)) \times \mathrm{H}^n(X, \mathscr{O}_X(-d-n-1)) \longrightarrow \mathrm{H}^n(X, \mathscr{O}_X(-n-1)) \xrightarrow{\sim} A$$

which is a perfect pairing of finitely generated free A-modules.

Proof. The main ingredient to finish the proof is the isomorphism (2.3.1) above. This allows us to identify $\mathrm{H}^i(X, \mathscr{O}_X(d))$ with $\check{\mathrm{H}}^i(\mathfrak{U}, \mathscr{O}_X(d))$, and these cohomology modules are displayed in (2.3.4). The only statement not deducible from (2.3.4) is (d) and this follows from the pairing in (1.2.8), which we showed is perfect right after we defined it. □

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