LECTURE 13

Date of Lecture: Nov 9, 2021

The symbol $\mathscr{A}b$ will denote the category of abelian groups and $\mathbb{S}\mathrm{ch}$ the category of schemes. If X is a topological space, $\mathscr{P}sh_X$ and $\mathscr{S}h_X$ denote the category of presheaves and the category of sheaves respectively on X. By a ring we mean a commutative ring with identity. For a ring A, Mod_A denotes the category of A-modules. For a sheaf of rings $\mathscr{A}$ on a topological space, $\mathrm{Mod}_{\mathscr{A}}$ will denote the category of $\mathscr{A}$-modules.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. **Fibre products**

1.1. ***S*-schemes.** Recall that if $S \in \mathbb{S}\mathrm{ch}$ then $\mathbb{S}\mathrm{ch}_{/S}$ is the category of scheme maps $u\colon X \to S$, with morphisms from $X \xrightarrow{u} S$ to $X' \xrightarrow{u'} S$ being scheme maps $h\colon X \to X'$ such that $u' \circ h = u$. We usually do not write $X \xrightarrow{u} S$ for an object of $\mathbb{S}\mathrm{ch}_{/S}$, preferring the simpler notation X, with the map u being understood. The map u is then called the "structure map" of the S-scheme X. Objects of $\mathbb{S}\mathrm{ch}_{/S}$ are called S-schemes, a term we just used. A morphism in $\mathbb{S}\mathrm{ch}_{/S}$ is often called an S-map or an S-morphism, or a map of S-schemes, or a morphism of S-schemes.

1.2. **Construction of fibre products.** Let S be a scheme and $X, Y \in \mathbb{S}\mathrm{ch}_{/S}$ with structure maps u and v respectively. The *fibre product* $X \times_S Y$ *of* X *and* Y *over* S is a triple (W, p, q) with $p\colon W \to X$ and $q\colon W \to Y$ being S-maps in such that if $f\colon T \to X$ and $g\colon T \to Y$ are S-maps then there is a unique map $(f, g)\colon T \to W$ such that $p \circ (f, g) = f$ and $q \circ (f, g) = g$. The situation is summarised by saying that if the solid arrows in the diagram below form a commutative diagram, then the broken arrow can be filled uniquely (so that the resulting diagram commutes).

(1.2.1)
$$\begin{tikzcd}
T \arrow[rrd, bend left, "g"] \arrow[rdd, bend right, "f"'] \arrow[rd, dashed, "\exists !", "{(f,g)}"'] & & \\
& W \arrow[r, "q"] \arrow[d, "p"'] & Y \arrow[d, "v"] \\
& X \arrow[r, "u"'] & S
\end{tikzcd}$$

From the universal property of $X \times_S Y$, it is clear that if it exists, it is unique up to unique isomorphism. We often write $X \times_S Y$ for W. In Problem 2 of HW 5 you were asked to show that if S, X and Y are affine, then $X \times_S Y$ exists. Now suppose S and Y are affine, and X arbitrary. Let U be an affine open subscheme of X, so that $U \times_S Y$ exists. It is not hard to see from the universal property of the various $U \times_S Y$, as U varies over affine open sub-schemes of X, that the $U \times_S Y$

glue to give a scheme W. Next let the data (T, f, g) be as above. Covering T by open subschemes of the form T_U with U affine open in X, and using the fact the universal property of fibre products, we see that $(f_U, g)|_{T_U \cap T_V} = (f_V, g)|_{T_U \cap T_V}$ for affine open subschemes U and V of X. These therefore glue to give a well defined map $\Psi \colon T \to W$ such that $p \circ \Psi = f$ and $q \circ \Psi = g$. That Ψ is the only map satisfying this relation is seen from the uniqueness of the various (f_U, g) as U varies over affine open subschemes of X. Thus (f, g) exists (it is Ψ), and is unique. We have proven that $X \times_S Y$ exists in the case where S and Y are affine. Now suppose Y is also an arbitrary S-scheme. Covering Y by affine opens, and using arguments identical to the one we just used, we see that $X \times_S Y$ exists in this case. Finally now assume S is also arbitrary. Cover S by affines opens and check that the schemes $u^{-1}(V) \times_V v^{-1}(V)$ glue as V varies over affine open subschemes of S, and the resulting scheme has the required universal property. From all this, it is clear that fibre products always exist.

The maps $p \colon W \to X$ and $q \colon W \to Y$, where W is the above fibre product, are called *projections*. Fibre product diagrams are often denoted as follows (note the small hollow square in the middle of the diagram):

(1.2.2)
$$\begin{tikzcd}
W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\
Y \arrow[r] & S
\end{tikzcd}$$

The above diagram is to be read as a commutative diagram of schemes such that W is the fibre product $X \times_S Y$ and the two arrows emanating from W are the projection maps. Such diagrams are called *cartesian squares* or *fibre product squares*, or sometimes, *cartesian diagrams*.

1.2.3. The scheme $X \times_S Y$ is a product in the category $\mathbb{S}\mathrm{ch}_{/S}$. We have shown that $\mathbb{S}\mathrm{ch}_{/S}$ has products. If you do not know the category theoretic definition of products, ignore this comment. Or look up `https://en.wikipedia.org/wiki/Product_(category_theory)`. In category theoretic terms $\begin{pmatrix} W \\ \downarrow \\ S \end{pmatrix} = \begin{pmatrix} X \\ \downarrow u \\ S \end{pmatrix} \times \begin{pmatrix} Y \\ \downarrow v \\ S \end{pmatrix}$.

1.3. **Gluing schemes and maps.** (Skip on first reading.) In the construction of fibre products we glued schemes and we glued maps, and we haven't quite discussed how this is done. Underlying that is the idea of gluing sheaves. So suppose X is a topological space, $\mathscr{X} = \{X_\alpha \mid \alpha \in \Lambda\}$ an open cover, and suppose further that for each $\alpha \in \Lambda$, $\mathscr{F}_\alpha$ a sheaf on X_α, and for each pair of indices $\alpha, \beta \in \Lambda$ we have isomorphisms $\varphi_{\alpha\beta} \colon \mathscr{F}_\beta|_{X_\alpha \cap X_\beta} \xrightarrow{\sim} \mathscr{F}_\alpha|_{X_\alpha \cap X_\beta}$ such that $\varphi_{\alpha\alpha} = \mathbf{1}$, $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$ for all $\alpha, \beta, \gamma \in \Lambda$. Then either by gluing étale spaces, or more elegantly (and in a way applicable to sheaves on general Grothendieck topologies) via the method indicated here, one can find a sheaf $\mathscr{F}$, unique up to unique isomorphism, such that we have isomorphisms $\psi_\alpha \colon \mathscr{F}|_{X_\alpha} \xrightarrow{\sim} \mathscr{F}_\alpha$ satisfying $\varphi_{\alpha\beta} \circ \psi_\beta|_{X_\alpha \cap X_\beta} = \psi_\alpha|_{X_\alpha \cap X_\beta}$.

Now suppose one has a standard gluing data on a collection of schemes $\{X_\alpha \mid \alpha \in \Lambda\}$. In other words, for every $(\alpha, \beta) \in \Lambda \times \Lambda$ we have an open subscheme $X_{\beta\alpha}$ of X_α and isomorphisms $\psi_{\alpha\beta} \colon X_{\alpha\beta} \xrightarrow{\sim} X_{\beta\alpha}$ satisfying, for $\alpha, \beta, \gamma \in \Lambda$, the relation $\psi_{\alpha\beta} \circ \psi_{\beta\gamma} = \psi_{\alpha\gamma}$ on $X_{\beta\gamma} \cap X_{\alpha\gamma}$. Standard topology tells us that the underlying topological spaces glue to give a topological space X, and in X the X_α can be regarded as open subsets, $X_{\alpha\beta}$ as $X_\alpha \cap X_\beta$, and $\psi_{\alpha\beta}$ as the identity map. This X together with these open sets is unique up to unique isomorphism. Now the $\mathscr{O}_{X_\alpha}$

can be glued on X by the previous paragraph to give us a sheaf of rings $\mathscr{O}_X$ on X. It is easy to see that $(X,\, \mathscr{O}_X)$ is a scheme, and is unique up to unique isomorphism in a way compatible with the gluing data.

Finally maps of schemes with gluing data can be glued as continuous maps and then upgraded to maps of locally ringed spaces. In slightly greater detail, suppose we have an open cover $\mathscr{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ of $X \in \mathbb{S}\mathrm{ch}$, and we write $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$ for $\alpha, \beta, \gamma \in \Lambda$. Suppose further that we have maps $f_\alpha \colon U_\alpha \to Y$ in $\mathbb{S}\mathrm{ch}$, one for each $\alpha \in \Lambda$, such that $f_\alpha|_{U_{\alpha\beta}} = f_\beta|_{U_{\alpha\beta}}$ for $\alpha\beta \in \Lambda$, then we have a continuous map $f \colon X \to Y$ obtained by gluing the f_α's. We also have maps of sheaves of rings $f_\alpha^\# \colon \mathscr{O}_Y \to (f_\alpha)_*\mathscr{O}_U$ for each $\alpha \in \Lambda$. Let $f_{\alpha\beta} := f_\alpha|_{U_{\alpha\beta}}$. By our hypotheses, $f_{\alpha\beta} = f_{\beta\alpha}$. The gluing condition implies that the composite $\mathscr{O}_Y \to (f_\alpha)_*\mathscr{O}_{U_\alpha} \to (f_{\alpha\beta})_*\mathscr{O}_{U_{\alpha\beta}}$ equals the composite $\mathscr{O}_Y \to (f_\beta)_*\mathscr{O}_{U_\beta} \to (f_{\alpha\beta})_*\mathscr{O}_{U_{\alpha\beta}}$. We have an exact sequence

$$0 \longrightarrow f_*\mathscr{O}_X \longrightarrow \prod_{\alpha \in \Lambda} (f_\alpha)_*\mathscr{O}_{U_\alpha} \longrightarrow \prod_{\alpha,\beta} (f_{\alpha\beta})_*\mathscr{O}_{U_{\alpha\beta}}$$

where the last arrow (over an open set V of Y) is given by

$$(\sigma_\alpha) \mapsto (\sigma_\alpha|_{f^{-1}(U_{\alpha\beta})\cap V} - \sigma_\beta|_{f^{-1}(U_{\alpha\beta})\cap V}).$$

This gives us a unique map $f^\# \colon \mathscr{O}_Y \to f_*\mathscr{O}_X$ of sheaves by the universal property of kernels. One checks that this is a map of sheaves of rings and induces a map $(f,\, f^\#) \colon X \to Y$ in $\mathbb{S}\mathrm{ch}$.

1.4. **Fibres of scheme maps.** Recall from commutative algebra that if $\phi \colon A \to B$ is a ring map, $f \colon X = \operatorname{Spec} B \longrightarrow \operatorname{Spec} A = Y$ the equivalent map of affine schemes, $\mathfrak{p}$ a prime ideal of A, then the fibre $f^{-1}(\mathfrak{p})$ is in bijective correspondence with the points of $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p}))$, where $\kappa(\mathfrak{p}) := A_\mathfrak{p}/(\mathfrak{p}A_\mathfrak{p})$ is the residue field of A at $\mathfrak{p}$. Indeed the primes of $B/(\mathfrak{p}B)$ are primes which contract to primes of A containing $\mathfrak{p}$, and on localising, the primes of $(B/(\mathfrak{p}B)_\mathfrak{p}$ contain primes which contract to precisely $\mathfrak{p}$. Now $B/(\mathfrak{p}B)_\mathfrak{p} = B_\mathfrak{p}/(\mathfrak{p}B_\mathfrak{p}) = B \otimes_A (A_\mathfrak{p}/(\mathfrak{p}A_\mathfrak{p})) = B \otimes_A \kappa(\mathfrak{p})$. The fibre $f^{-1}(\mathfrak{p})$ is given the scheme structure of $\operatorname{Spec}(B \otimes_A \kappa(\mathfrak{p})) = X \times_Y \operatorname{Spec} \kappa(\mathfrak{p})$.

A little thought shows that if one has a map $f \colon X \to Y$ of schemes, and $y \in Y$ is a point, then $f^{-1}(y)$ is in bijective correspondence with the points of $X \times_Y \operatorname{Spec} \kappa(y)$ where $\kappa(x) = \mathscr{O}_{Y,y}/\mathfrak{m}_y$ is the residue field of $\mathscr{O}_Y$ at y. The fibre $f^{-1}(y)$ is given the scheme structure of $X \times_Y \operatorname{Spec} \kappa(y)$. We therefore have a cartesian square

(1.4.1)
$$\begin{tikzcd}
f^{-1}(y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
\operatorname{Spec} \kappa(y) \arrow[r, "y"'] & Y
\end{tikzcd}$$

Note that we labelled the bottom horizontal arrow as y. This is not uncommon in Algebraic Geometry where we often think of points as arrows. This is a small part of the major overhaul in thinking Grothendieck initiated.

1.4.2. If $S = \operatorname{Spec} R$, we often write $X \times_R Y$ instead of $X \times_S Y$. Similarly, in this case $\mathbb{S}\mathrm{ch}_{/S}$ is often written as $\mathbb{S}\mathrm{ch}_{/R}$. Finally, note that $\mathbb{S}\mathrm{ch} = \mathbb{S}\mathrm{ch}_{/\mathbf{Z}}$ where $\mathbf{Z}$ is the ring of integers. This is because every ring is a $\mathbf{Z}$ algebra in a canonical way, and if $X \in \mathbb{S}\mathrm{ch}$, then the maps $U \to \operatorname{Spec} \mathbf{Z}$ as U varies over affine open subschemes of X glue to give a canonical morphism $X \to \operatorname{Spec} Z$. A slicker way is to invoke

Theorem 1.1.2 of Lecture 9 and use the fact that we have a canonical ring map $\mathbf{Z} \to \Gamma(X, \mathscr{O}_X)$ so that by *loc.cit.* we have a canonical map $X \to \operatorname{Spec} \mathbf{Z}$.

2. Separatedness

2.1. **The diagonal map.** Let $X \to Y$ be a scheme map. Let, as usual $\mathbf{1} = \mathbf{1}_X$ denote the identity map on X. Let $\Delta_{X/Y} \colon X \to X \times_Y X$ be the map $\Delta_{X/Y} = (\mathbf{1}, \mathbf{1})$ where we are using the notation used in Diagram (1.2.1) with $(f, g) = (\mathbf{1}, \mathbf{1})$. The map $\Delta_{X/Y}$ is called the *diagonal morphism of the Y-scheme X*.

If $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ and $\phi \colon A \to B$ induces $X \to Y$, then $\Delta_{X/Y}$ is, in terms of ring homomorphisms, equivalent to the map of rings $B \otimes_A B \to B$ given by $b_1 \otimes b_2 \mapsto b_1 b_2$. Since this map is surjective (indeed, if $b \in B$, then $b \otimes 1 \mapsto b$), in this case the map $\Delta_{X/S}$ is a *closed immersion*.

In the general case the diagonal map need not be a closed immersion, but it is always an *immersion*, i.e. it is a closed immersion followed by an open immersion. To see this, let V be an affine open subscheme of Y, and U an affine open subscheme of X. From the "construction" of $X \times_Y X$, we see that $U \times_V U$ is open in $X \times_Y X$ (one should prove it by using universal properties, but this was a quick fix to get on with the story). It is easy to see that $\Delta^{-1}_{X/Y}(U \times_V U) = U$ and the restriction of $\Delta_{X/Y}$ to U is $\Delta_{U/V}$, i.e. the diagram

$$\begin{tikzcd}
U \arrow[r, "\Delta_{U/V}"] \arrow[d, hook] \arrow[dr, phantom, "\square"] & U \times_V U \arrow[d, hook] \\
X \arrow[r, "\Delta_{X/Y}"'] & X \times_Y X
\end{tikzcd}$$

is cartesian. The horizontal arrow on the top is a closed immersion since all schemes there are affine. Let $W = \bigcup (U \times_V U)$ where the union is taken over U and V as above. Then W is an open subscheme of $X \times_Y X$, $\Delta_{X/Y}$ factors as $X \xrightarrow{\Delta'} W \subset X \times_Y X$. The first map Δ' is a closed immersion by looking at the cartesian diagram above. The second is an open immersion.

Definition 2.1.1. A map $f \colon X \to Y$ in $\mathbb{S}\mathrm{ch}$ is said to be *separated* if $\Delta X/Y$ is a closed immersion. A scheme X is said to be *separated* if the natural map $X \to \operatorname{Spec} \mathbf{Z}$ is separated, where $\mathbf{Z}$ is the ring of integers.

2.1.2. If $Y = \operatorname{Spec} R$, one often writes $\Delta_{X/R}$ instead of $\Delta_{X/Y}$.

2.1.3. Let $S \in \mathbb{S}\mathrm{ch}$ and $X \in \mathbb{S}\mathrm{ch}_{/S}$. From the universal property of fibre products, if U and V are open subschemes of a scheme X, then we have a natural map $U \cap V \to U \times_S V$. The map is (i, j) where $i \colon U \cap V \to U$ and $j \colon U \cap V \to V$ are the natural inclusions. Further, it is clear that $\Delta^{-1}_{X/S}(U \rightarrow, es_S V) = U \cap V$ and (i, j) is simply the restriction of $\Delta_{X/S}$ to $U \cap V$.

Proposition 2.1.4. *Let U and V be affine open subschemes of a separated scheme X. Then $U \cap V$ is affine.*

Proof. We have $U \cap V = \Delta^{-1}_{X/\mathbf{Z}}(U \times_{\mathbf{Z}} V)$. Since $\Delta_{X/\mathbf{Z}}$ is a closed immersion (X being separated), this means $U \cap V$ is a closed subscheme of $U \times_{\mathbf{Z}} V$. Now $U \times_{\mathbf{Z}} V$ is affine, and all closed subschemes of affine schemes are affine, and hence $U \cap V$ is affine. $\square$

Proposition 2.1.5. *Let $X \in \mathbb{S}ch$ and suppose there is a cover $\mathfrak{U}$ of X by affine open subschemes such that for every $U, V \in \mathfrak{V}$, the natural map $U \cap V \to U \times_{\mathbf{Z}} V$ of §2.1.3 is a closed immersion. Then X is separated.*

Proof. The open subschemes $U \times_{\mathbf{Z}} V$ cover $X \times_{\mathbf{Z}} X$ as U and V vary over affine open subschemes of X. From the hypothesis of the proposition and from the observations made in § 2.1.3, it is clear that the map

$$\Delta^{-1}_{X/\mathbf{Z}}(U \times_{\mathbf{Z}} V) \xrightarrow{\text{via } \Delta_{X/\mathbf{Z}}} U \times_{\mathbf{Z}} V$$

is a closed immersion for every pair of affine open subschemes of X. It follows that $\Delta_{X/\mathbf{Z}}$ is a closed immersion. ☐

Theorem 2.1.6. *Let $S = \bigoplus_n S_n$ be a graded ring. Then* $\mathrm{Proj}(S)$ *is separated.*

Proof. As always, let $S_+ = \bigoplus_{n>0} S_n$. Recall that for f, g homogeneous elements of S_+, we can identify $D_+(f)$ with $\mathrm{Spec}\, S_{(f)}$ and that $D_+(fg) = D_+(f) \cap D_+(g)$. Since the $D_+(f)$ cover X as f varies over homogeneous elements of S_+, according to Proposition 2.1.5, we only have to check that the natural map $S_{(f)} \otimes_{\mathbf{Z}} S_{(g)} \to S_{(fg)}$ is surjective. This is an easy exercise. ☐

3. Čech cohomology and derived functor cohomology

3.1. **Affine morphisms.** A morphism of schemes $f\colon X \to Y$ is said to be an *affine morphism* if $f^{-1}(U)$ is an scheme for every affine open subsceheme U of Y.

An affine map is clearly quasi-compact.[1] Indeed, suppose V is quasi-compact in Y. Then V can be covered by a finite number of affine opens $U_1, \dots, U_n$. Now $f^{-1}(V) = \bigcup_{i=1}^n f^{-1}(U_i)$. Since $f^{-1}(U_i)$ are affine, they are quasi-compact, whence $f^{-1}(V)$ is the finite union of quasi-compact open sets. It follows that it is quasi-compact.

3.1.1. Here are some examples and properties of affine maps.

1. Let $j\colon U \subset X$ be the open immersion of an affine open subscheme U of a *separated* scheme X. For every affine open subscheme V of X we have $j^{-1}(V) = U \cap V$, which is affine according to Proposition 2.1.4. It follows that the map i is an affine morphism.

2. Let $\phi\colon A \to B$ be a map of rings, $X = \mathrm{Spec}\, B$, $Y = \mathrm{Spec}\, A$, and $\Psi\colon X \to Y$ the map equivalent to ϕ. Then Ψ is an affine map. To see this, first observe for $f \in A$, $\Psi^{-1}(D_A(f)) = D_B(\phi(f))$, where the subscripts A and B on the classical open sets $D(f)$ and $D(\phi(f))$ have been introduced for good book-keeping purposes. Next note that Ψ is a quasi-compact map, for if $V \subset Y$ is quasi-compact and open, then V can be covered by a finite number of $D_A(f)$, say by $D_A(f_1), \dots, D_A(f_n)$. Now $\Psi^{-1}(V) = \bigcup_{i=1}^n D_B(\phi(f_i))$, and hence $\Psi^{-1}(V)$ is quasi-compact. Now suppose this V is an affine open subscheme of Y. Let $\bar{f_i} = f_i|_V$. Then $\bar{f_1}, \dots, \bar{f_n}$ generate the unit ideal of $\Gamma(V, \mathscr{O}_Y)$. Let $U = \Psi^{-1}(V)$. Then U is quasi-compact since Ψ is a quasi-compact map. Let $\phi_i = \phi(f_i)$ and $\bar{\phi}_i = \phi_i|_U$, $i = 1, \dots, n$. Then $\bar{\phi}_1, \dots, \bar{\phi}_n$ generate the unit ideal in $\Gamma(U, \mathscr{O}_X)$. Now, $U_{\bar{\phi}_i} = \Psi^{-1}(D_A(f_i)) = D_B(\phi_i)$, $i = 1, \dots, n$. Thus $U_{\bar{\phi}_i}$ is affine and $\bar{\phi}_1, \dots, \bar{\phi}_n$ generate the unit ideal in $\Gamma(U, \mathscr{O}_X)$. From Proposition 1.4.2 of Lecture 12, it follows that U is affine.

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[1] i.e. the inverse image of a quasi-compact open subscheme is quasi-compact.

3. If a map of schemes $\Psi\colon X \to Y$ is such that Y has an open cover $\mathfrak{V}$ consisting of affine opens, such that $\Psi^{-1}(V)$ is affine for $V \in \mathfrak{V}$, then Ψ is affine. We can see this as follows. Let $\mathfrak{V} = \{V_\alpha\}$. For V open in Y, let Ψ_V denote the map $\Psi^{-1}(V) \to V$ induced by Ψ. By **2** above, Ψ_{V_α} is quasi-compact for every index α. We first claim that Ψ is quasi-compact. To that end suppose W is quasi-compact. For each index α, ;et $W_\alpha = W \cap V_\alpha$. If V is an affine open subset of W_α, then since Ψ_{V_α} is affine, we have $\Psi^{-1}(V)$ is affine. Since each W_α can be covered by affine opens, and the W_α cover W, we can find an open cover $\{D_i\}$ of W such that the D_i are affine, and for each i there is an index $\alpha(i)$ such that $D_i \subset W_{\alpha(i)}$. Since W is quasi-compact, we may, without loss of generality assume that $\{D_i\}$ is a finite cover of W. Thus $\Psi^{-1}(W) = \bigcup_i \Psi^{-1}(D_i)$ and hence $\psi^{-1}(W)$ is a finite union of open affine open subschemes of X. It follows that $\Psi^{-1}(W)$ is quasi-compact. To show Ψ is affine we plan to apply Proposition 1.4.2 of Lecture 12 as we did in **2**. So suppose the quasi-compact W we were with is affine. Then, arguing as we did moments ago, we can find $f_1, \dots, f_n \in R = \Gamma(W, \mathscr{O}_Y)$ such that the f_i's generate the unit ideal in R, and there exist indices $\alpha(i)$ such that $D_R(f_i) \subset W_{\alpha(i)}$ for $i = 1, \dots, n$. Let $U = \Psi^{-1}(W)$, $S = \Gamma(U, \mathscr{O}_X)$ and ϕ_i the image of f_i in S for $i = 1, \dots, n$ under the natural map $R \to S$. Then $U_{\phi_i} = \Psi^{-1}(D_R(f_i))$ is affine for $i = 1, \dots, n$ and the ϕ_i generate the unit ideal of S. Since U is quasi-compact, it follows from *loc.cit.* that U is affine, proving that Ψ is affine.

4. If $\Psi\colon X \to Y$ is an affine map, then $\Psi_*\mathscr{F}$ is quasi-coherent on Y whenever $\mathscr{F}$ is quasi-coherent on X. To see this, we can reduce to the case where Y is affine, say $Y = \operatorname{Spec} A$. Then X is also affine, say $\operatorname{Spec} B$. The assertion follows from the fact that if $M \in \mathrm{Mod}_B$, then $\Psi_*(\widetilde{M}_B) = \widetilde{M}_A$.

In the following theorem, for $Z \in \mathbb{S}\mathrm{ch}$, the category of quasi-coherent $\mathscr{O}_Z$-modules is denoted as $\mathrm{Qcoh}(Z)$. This is what we denoted by the symbol Z_{qc} in earlier lectures.

Theorem 3.1.2. *Let $f\colon X \to Y$ be an affine map and $\mathscr{F} \in \mathrm{Qcoh}(X)$. Then*

(a) *The functor $f_*\colon \mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$ is exact.*
(b) $\mathrm{R}^i f_*\mathscr{F} = 0$ *for $i \geq 1$.*
(c) *We have canonical isomorphisms*

$$\mathrm{H}^i(Y,\, f_*\mathscr{F}) \xrightarrow{\sim} \mathrm{H}^i(X,\, \mathscr{F})$$

for $i \geq 0$.

Proof. For (a) we reduce immediately to the case where X and Y are affine and induced by a ring map $A \to B$. The assertion follows from the fact that is we have an exact sequence of B-modules $0 \to M' \to M \to M'' \to 0$, then it is an exact sequence of A-modules.

For parts (b) and (c) we fix a classical flasque resolution $\mathscr{F} \to \mathscr{G}^\bullet$ of $\mathscr{F}$. Let $y \in Y$ and V an affine open neighbourhood of y. Let $U = f^{-1}(V)$. Then U is affine, and hence $\mathrm{H}^i(U, \mathscr{F}) = 0$ for $i \geq 1$ by Serre's theorem (Cartan's Theorem B). In other words

$$0 \longrightarrow \Gamma(U,\, \mathscr{F}) \longrightarrow \Gamma(U,\, \mathscr{G}^0) \longrightarrow \dots \longrightarrow \Gamma(U,\, \mathscr{G}^n) \longrightarrow \dots$$

is exact. Since $\Gamma(U, \mathscr{H}) = \Gamma(V, f_*\mathscr{H})$ for any $\mathscr{H} \in \mathcal{S}\!h_X$, it follows that the sequence

$$0 \longrightarrow \Gamma(V, f_*\mathscr{F}) \longrightarrow \Gamma(V, f_*\mathscr{G}^0) \longrightarrow \dots \longrightarrow \Gamma(V, f_*\mathscr{G}^n) \longrightarrow \dots$$

is exact. Since V is an arbitrary affine open neighbourhood of y, it follows that $(f_*\mathscr{F})_y \to (f_*\mathscr{G}^\bullet)_y$ is a classical resolution. Since this is true for all $y \in Y$, it follows that $f_*\mathscr{F} \to f_*\mathscr{G}^\bullet$ is a classical resolution. Now the direct image of a flasque sheaf is flasque. Hence $f_*\mathscr{G}^\bullet$ is a flasque resolution of $f_*\mathscr{F}$. Since flasque sheaves are f_*-acyclic (see Remark 3.1.3 after the proof), (b) follows by the abstract de Rham theorem (see Theorem 1.1.6 of Lectures 10 and 11).

This more or less gives part (c) too. Flasque sheaves on a topological space Z are $\Gamma(Z, -)$-acyclic. Using the abstract de Rham theorem twice we have

$$\begin{aligned} \mathrm{H}^i(Y, f_*\mathscr{F}) &\xrightarrow{\sim} \mathrm{H}^i(\Gamma(Y, f_*\mathscr{G}^\bullet)) \\ &=\!=\!= \mathrm{H}^i(\Gamma(X, \mathscr{G}^\bullet)) \\ &\xrightarrow{\sim} \mathrm{H}^i(X, \mathscr{F}), \end{aligned}$$

giving (c). □

3.1.3. In the proof above, we used the fact that flasque sheaves are f_*-acyclic. This is seen as follows. Let $\mathscr{G}$ be flasque on X and $\mathscr{G} \to \mathscr{E}^\bullet$ a bounded below injective resolution. Let V be open in Y. Then $\mathscr{G}|_{f^{-1}(V)} \to \mathscr{E}^\bullet|_{f^{-1}(V)}$ is a bounded below injective resolution. It follows that $\Gamma(f^{-1}(V), \mathscr{G}) \to \Gamma(f^{-1}(V), \mathscr{E}^\bullet)$ is a quasi-isomophism, since $\mathscr{G}|_{f^{-1}(V)}$ is an acyclic sheaf. This quasi-isomorphism shows that $f_*\mathscr{G} \to f_*\mathscr{E}^\bullet$ is a quasi-isomorphism. It follows that $\mathrm{R}^i f_*\mathscr{G} = 0$ for $i \geq 1$ since $f_*\mathscr{G}$ is concentrated (as a complex) in degree 0, whence $\mathrm{H}^i(f_*\mathscr{G}) = 0$ for $i \geq 1$.

Corollary 3.1.4. *Let X be a separated scheme, U an affine open subscheme of X, $j\colon U \to X$ the resulting open inclusion, and $\mathscr{F}$ a quasi-coherent sheaf on U. Then $j_*\mathscr{F}$ is acyclic.*

Proof. This is an immediate consequence of item **1** of 3.1.1, part (c) of Theorem 3.1.2, and Serre's Theorem (Cartan's Theorem B). □

3.1.5. Suppose $\{E_i\}_{i\in I}$ is a collection of injective objects in an abelian category $\mathscr{A}$ which allows arbitrary products. Suppose A and B are objects in $\mathscr{A}$ with A a subobject of B, and suppose we have a map $\phi\colon A \to \prod_{i\in I} E_i$. Then we have maps $\phi_i\colon A \to E_i$ for each $i \in I$, and each of these extends to a map $\psi_i\colon B \to E_i$, since E_i are injective. It follows that we have a map $\psi = \prod_{i\in I}\psi_i\colon B \to \prod_{i\in I} E_i$. In other words $\prod_{i\in I} E_i$ is also injective. One consequence is that if $\mathscr{A}$ has enough injectives, and allows arbitrary products, and if $\mathscr{B}$ is another abelian category allowing arbitrary products, $T\colon \mathscr{A} \to \mathscr{B}$ left exact and commuting with direct products, then $\mathrm{R}^nT(\prod_{i\in I} A_i) = \prod_{i\in I}\mathrm{R}^nT(A_i)$, for any collection of objects $\{A_i \mid i \in I\}$ in $\mathscr{A}$. Indeed if $0 \to A_i \to E_i^\bullet$ is a classical injective resolution of A_i then $0 \to \prod_i A_i \to \prod_i E_i^\bullet$ is an injective resolution. The assertion follows easily. This is useful when we compare Čech cohomology with the derived functor cohomology.

3.2. **The Čech cohomology of quasi-coherent sheaves on separated schemes.**

Recall that if X is a topological space, $\mathfrak{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ an open cover with Λ well-ordered, then for $\mathscr{F} \in \mathcal{S}\!h_X$ we have a resolution

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \dots \longrightarrow \mathscr{C}^n(\mathfrak{U}, \mathscr{F}) \longrightarrow \dots \tag{3.2.1}$$

of $\mathscr{F}$ by the sheaf Čech complex $\mathscr{C}^\bullet(\mathfrak{U},\,\mathscr{F})$ (see Problem 2 of HW 2). For a p-tuple of indices $\boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_p)$, if we write $U_{\alpha_0\dots\alpha_p}$ for $U_{\alpha_0} \cap \dots \cap U_{\alpha_p}$, $j^{\boldsymbol{\alpha}} \colon U_{\alpha_0\dots\alpha_p} \to X$ for the resulting open inclusion, and $\mathscr{F}_{\alpha_0\dots\alpha_p}$ for the restriction of $\mathscr{F}$ to $U_{\alpha_0\dots\alpha_p}$, then clearly

$$\mathscr{C}^p(\mathfrak{U},\,\mathscr{F}) = \prod_{\alpha_0 < \dots < \alpha_p} j^{\boldsymbol{\alpha}}_* \mathscr{F}_{\alpha_0\dots\alpha_p}. \tag{3.2.2}$$

Suppose $0 \to \mathscr{F} \to \mathscr{E}^\bullet$ is an injective resolution of $\mathscr{F}$. Since (3.2.1) is a resolution of $\mathscr{F}$ we have a homotopy unique map $\theta \colon \mathscr{C}^\bullet \to \mathscr{E}^\bullet$ lifting the identity map on $\mathscr{F}$. Applying the composite functor $\mathrm{H}^i \circ \Gamma(X, -)$ to θ we get well defined maps, one for each i,

$$\check{\mathrm{H}}^i(\mathfrak{U},\,\mathscr{F}) \longrightarrow \mathrm{H}^i(X,\,\mathscr{F}). \tag{3.2.3}$$

We are using the fact that $\Gamma(X,\,\mathscr{C}^\bullet(\mathfrak{U},\,-)) = C^\bullet(\mathfrak{U},\,-)$. The maps in (3.2.3) are the so called *Čech to derived functor maps*, or the *comparison maps* from Čech cohomology to derived functor cohomology.

Theorem 3.2.4. *Let X be a separated scheme and $\mathfrak{U} = \{U_\alpha \mid \alpha \in \Lambda\}$ an affine open cover, with the index set Λ a well-ordered set. Let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then the comparison map*

$$\check{\mathrm{H}}^i(\mathfrak{U},\,\mathscr{F}) \longrightarrow \mathrm{H}^i(X,\,\mathscr{F})$$

of (3.2.3) is an isomorphism for every $i \geq 0$.

Proof. According to (3.2.2) and Corollary 3.1.4, $\mathscr{C}^p(\mathfrak{U},\,\mathscr{F})$ is the product of acyclic sheaves and hence by Remark 3.1.5, it is acyclic. The result follows from the abstract de Rham's theorem, i.e. Theorem 1.1.6 of Lectures 10 and 11. ☐

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