LECTURE 12

Date of Lecture: Nov 2, 2021

The symbol $\mathcal{A}b$ will denote the category of abelian groups and $\mathbb{S}\mathrm{ch}$ the category of schemes. If X is a topological space, $\mathcal{P}sh_X$ and $\mathcal{S}h_X$ denote the category of presheaves and the category of sheaves respectively on X. By a ring we mean a commutative ring with identity. For a ring A, Mod_A denotes the category of A-modules. For a sheaf of rings $\mathscr{A}$ on a topological space, $\mathrm{Mod}_{\mathscr{A}}$ will denote the category of $\mathscr{A}$-modules.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. Cohomology of quasi-coherent sheaves on affine schemes

We are following [K] in this section.

1.1. Closed immersions and other preliminaries. We remind the reader that a closed immersion $Z \hookrightarrow X$ of schemes is a map such that as a topological space Z is a closed subspace of X, and its scheme structure is given by a quasi-coherent ideal sheaf $\mathscr{I}$, i.e. for an affine open $U = \operatorname{Spec} R$ in X $U \cap Z$ is an open set of Z supporting an affine scheme, and the scheme structure on $U \cap Z$ is R/I where $I = \Gamma(U, \mathscr{I})$.

We also implicitly use the results of [AM, Chapter 1, Exercise 15], in the following sense. All the results of this exercise are compatible with formation of localisations A_f of a ring A, and hence have a scheme theoretic generalisation. For example if $\mathscr{I}$ and $\mathscr{J}$ are quasi-coherent ideal sheaves on a scheme X, and Z and Z' the closed subschemes of X corresponding to $\mathscr{I}$ and $\mathscr{J}$, then $Z \cup Z'$ and $Z \cap Z'$ are the schemes defined by $\mathscr{I} \cap \mathscr{J}$ and $\mathscr{I} + \mathscr{J}$ respectively.

In particular, since the nilradical of a ring R behaves well with the formation of localisations R_f, it makes sense to to talk about the nilradical ideal sheaf of a scheme X and also of the radical $\sqrt{\mathscr{I}}$ of a quasi-coherent ideal sheaf $\mathscr{I}$, and hence of the reduced structure X_{red} of a scheme X. Note that given a closed subset Z of a scheme X, it has many scheme structures on it (if the quasi-coherent ideal sheaf $\mathscr{I}$ gives one, then so do the ideals $\mathscr{I}^n$ or any quasi-coherent ideal sheaf $\mathscr{J}$ such that $\sqrt{\mathscr{I}} = \sqrt{\mathscr{J}}$). The various quasi-coherent ideals giving a scheme on Z have the same radical, and hence there is a unique reduced structure on Z, given by a quasi-coherent radical ideal sheaf.

1.2. Topological preliminaries. Fix a topological space X. For U an open set, and $\mathscr{F} \in \mathcal{S}h_X$, as in (5.1.1) of Lectures 10 and 11, define ${}_U\mathscr{F} \in \mathcal{S}h_X$ by the formula ${}_U\mathscr{F} = i_*(\mathscr{F}|_U)$, where i is the inclusion $U \subset X$. Equivalently ${}_U\mathscr{F}$ is the sheaf $V \mapsto \mathscr{F}(U \cap V)$ with obvious restriction maps (see (5.1.2) of *loc.cit.*). It is clear, and recorded in Proposition 5.1.4 of *loc.cit.*, that if $\mathscr{F}$ is flasque, so is ${}_U\mathscr{F}$. The formula ${}_U\mathscr{F}(V) = \mathscr{F}(U \cap V)$ shows (as in (5.1.3) of *loc.cit.*) that there is a natural map of sheaves

$$\mathscr{F} \longrightarrow {}_U\mathscr{F}. \tag{1.2.1}$$

The following statement and proof is taken from [K], a highly readable paper.

Proposition 1.2.2. *Let $\mathcal{B}$ be a basis for the topology on X which is closed under finite intersections. Fix $n \geq 1$. Suppose $\mathscr{F}$ is a sheaf on X such that $\mathrm{H}^i(X,\, {}_U\mathscr{F}) = 0$ for $0 < i < n$ and $U \in \mathcal{B}$. Let $\alpha \in \mathrm{H}^n(X,\, \mathscr{F})$. Then there exists an open cover $\mathfrak{V}$ of X, with $\mathfrak{V} \subset \mathcal{B}$, such that the image of α in $\mathrm{H}^n(X,\, {}_V\mathscr{F})$, induced by (1.2.1), is zero for every $V \in \mathfrak{V}$.*

Proof. Note that when $n = 1$ the condition $0 < i < n$ is vacuous and hence every sheaf satisfies the hypothesis of the proposition. The result follows in this case from Lemma 5.1.5 of Lectures 10 and 11.

Assume now that $n > 1$. Note that $\mathrm{H}^1(U,\, \mathscr{F}) = 0$ for all $U \in \mathcal{B}$. Let $\mathscr{G}$ be a flasque sheaf containing $\mathscr{F}$ and set $\mathscr{H} = \mathscr{G}/\mathscr{F} := \mathrm{coker}\,(\mathscr{F} \hookrightarrow \mathscr{G})$. We have an exact sequence

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow \mathscr{H} \longrightarrow 0. \tag{$*$}$$

If $U \in \mathcal{B}$ then applying $\Gamma(U,\, -)$ to the above sequence and using the fact that that $\mathrm{H}^1(U,\, \mathscr{F}) = 0$, we see that $\Gamma(U,\, \mathscr{G}) \to \Gamma(U,\, \mathscr{H})$ is surjective. In particular, if V and W are members of $\mathcal{B}$, then as $V \cap W \in \mathcal{B}$, we have $\mathscr{G}(V \cap W) \to \mathscr{H}(V \cap W)$ is surjective, which means that for fixed $V \in \mathcal{B}$, ${}_V\mathscr{G}(W) \to {}_V\mathscr{H}(W)$ is surjective for all $W \in \mathcal{B}$. In other words ${}_V\mathscr{G} \to {}_V\mathscr{H}$ is an epimorphism of sheaves (since $\mathcal{B}$ is a basis and around each point we can find arbitrarily small neighbourhoods of from $\mathcal{B}$) giving us an exact sequence of sheaves

$$0 \longrightarrow {}_V\mathscr{F} \longrightarrow {}_V\mathscr{G} \longrightarrow {}_V\mathscr{H} \longrightarrow 0 \qquad (V \in \mathcal{B}). \tag{$\dagger$}$$

Now for any open set U in X, $\mathscr{G}|_U$ is flasque, therefore acyclic, whence

$$\mathrm{H}^i(U,\, \mathscr{G}) = \mathrm{H}^{i+1}(U,\, \mathscr{G}) = 0 \quad (i \geq 1).$$

Thus by the long exact sequence in cohomology associated to $(*)$ we have

$$\mathrm{H}^i(U,\, \mathscr{H}) \xrightarrow{\ \sim\ } \mathrm{H}^{i+1}(U,\, \mathscr{F}), \quad (i \geq 1) \tag{$\flat$}$$

for every open set U in X. Picking U in $\mathcal{B}$ and using the hypothesis on $\mathscr{F}$ we see that the isomorphism $(\flat)$ gives

$$\mathrm{H}^i(U,\, \mathscr{H}) = 0 \qquad (0 < i < n-1).$$

Thus $\mathscr{H}$ satisfies the hypotheses of the proposition for $n-1$. By induction we deduce that given $\beta \in \mathrm{H}^{n-1}(X,\, \mathscr{H})$ we have $\mathfrak{V} \subset \mathcal{B}$ with $X = \bigcup_{V \in \mathfrak{V}} V$ such that the image of β in $\mathrm{H}^{n-1}(X,\, {}_V\mathscr{H})$ is zero for every $V \in \mathfrak{V}$. Let β be the unique element mapping to $\alpha \in \mathrm{H}^n(X,\, \mathscr{F})$ under the isomorphism $\mathrm{H}^{n-1}(X,\, \mathscr{H}) \xrightarrow{\ \sim\ } \mathrm{H}^n(X,\, \mathscr{F})$ in $(\flat)$, with $U = X$ and $i = n-1$. Let $\mathfrak{V} \subset \mathcal{B}$ be the open cover associated to β just described. For $V \in \mathfrak{V}$, the exact sequences $(*)$ and $(\dagger)$ fit into the commutative diagram below with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{H} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle (1.2.1)} & & \downarrow{\scriptstyle (1.2.1)} & & \downarrow{\scriptstyle (1.2.1)} & & \\
0 & \longrightarrow & {}_V\mathscr{F} & \longrightarrow & {}_V\mathscr{G} & \longrightarrow & {}_V\mathscr{H} & \longrightarrow & 0
\end{array}$$

We therefore have a commutative diagram (by the naturality of connecting homomorphisms)

$$\begin{array}{ccc} \mathrm{H}^{n-1}(X,\,\mathscr{H}) & \xrightarrow{\sim} & \mathrm{H}^{n}(X,\,\mathscr{F}) \\ \downarrow{\scriptstyle (1.2.1)} & & \downarrow{\scriptstyle (1.2.1)} \\ \mathrm{H}^{n-1}(X,\,{}_V\mathscr{H}) & \xrightarrow{\sim} & \mathrm{H}^{n}(X,\,{}_V\mathscr{F}) \end{array}$$

Since β maps to zero under the downward arrow on the left, and β maps to α under the horizontal isomorphism on the top row, we are done. □

1.3. **Serre's theorem.** The main theorem of this subsection, and one of the building blocks of scheme-theoretic algebraic geometry, is the following famous theorem of Serre (the proof we give from [K] is due to G. R. Kempf):

Theorem 1.3.1. (Serre) *Let X be an affine scheme. Then all quasi-coherent $\mathscr{O}_X$-modules are acyclic, i.e. $\mathrm{H}^n(X,\,\mathscr{F}) = 0$ for all $n \geq 1$ and all $\mathscr{F} \in X_{qc}$.*

Proof. Let $A = \Gamma(X,\,\mathscr{O}_X)$ so that $X = \operatorname{Spec} A$. We will apply Proposition 1.2.2 with $\mathcal{B} = \{D(f) \mid f \in A\}$. First let us prove the theorem for $n = 1$. To that end, let $\mathscr{F}$ be a quasi-coherent $\mathscr{O}_X$-module. Then $\mathscr{F} = \widetilde{M}$ where $M = \Gamma(X,\,\mathscr{F})$. Let $\alpha \in \mathrm{H}^1(X,\,\mathscr{F})$. By Proposition 1.2.2, and the quasi-compactness of affine schemes, we have $f_1, \ldots, f_d \in A$ such that $X = \cup_{i=1}^{d} D(f_i)$ and such that the image of α in $\mathrm{H}^1(X,\,{}_{D(f_i)}\mathscr{F})$ is zero for $i = 1, \ldots, d$. Since $M \to \bigoplus_{i=1}^{d} M_{f_i}$ is injective and since clearly $\widetilde{M_{f_i}} = {}_{D(f_i)}\mathscr{F}$, we have an inclusion $\mathscr{F} \hookrightarrow \bigoplus_{i=1}^{d} {}_{D(f_i)}\mathscr{F}$. With $\mathscr{C}$ the cokernel of this inclusion, we have an exact sequence of quasi-coherent sheaves

$$(\#) \qquad 0 \longrightarrow \mathscr{F} \longrightarrow \bigoplus_{i=1}^{d} {}_{D(f_i)}\mathscr{F} \longrightarrow \mathscr{C} \longrightarrow 0.$$

Since X is affine, $\Gamma(X,\,-)$ is exact on quasi-coherent sheaves. In particular the complex $\Gamma(X,\,(\#))$ is exact. Thus we have an an inclusion (using the long exact sequence associated to (#))

$$\mathrm{H}^1(X, \mathscr{F}) \hookrightarrow \bigoplus_{i=1}^{d} \mathrm{H}^1(X,\,{}_{D(f_i)}\mathscr{F}).$$

Now the image of α under this inclusion is zero by our choice of the cover $\{D(f_i) \mid i = 1, \ldots, d\}$. Thus $\alpha = 0$ and hence $\mathrm{H}^1(X\,\mathscr{F}) = 0$.

Suppose now that $n > 1$ and that for every affine scheme Z and every quasi-coherent sheaf $\mathscr{H}$ on Z, $\mathrm{H}^i(Z,\,\mathscr{H}) = 0$ for $0 < i < n$. Let $\mathscr{F}$ be a quasi-coherent sheaf on our scheme X. Since the various $D(f)$, for $f \in A$, are all affine, therefore by our induction assumption, $\mathrm{H}^i(D(f),\,\mathscr{F})$ vanishes for $0 < i < n$. Let $\alpha \in \mathrm{H}^n(X,\,\mathscr{F})$. Once again apply Proposition 1.2.2 to $\mathscr{F}$ with $\mathcal{B} = \{D(f) \mid f \in A\}$. This time the conclusion is that we have $f_1, \ldots, f_d \in A$ such that $X = \bigcup_{i=1}^{d} D(f_i)$ and such that the image of α in $\mathrm{H}^n(X,\,{}_{D(f_i)}\mathscr{F})$ is zero for $i = 1, \ldots, d$. Consider again the exact sequence (#) associated with this cover of X. By induction $\mathrm{H}^{n-1}(X,\,\mathscr{C}) = 0$ since $\mathscr{C}$ is quasi-coherent. Using the long exact sequence in cohomology associated with (#) we have an inclusion

$$\mathrm{H}^n(X, \mathscr{F}) \hookrightarrow \bigoplus_{i=1}^{d} \mathrm{H}^n(X,\,{}_{D(f_i)}\mathscr{F}).$$

Since the image of α under this inclusion is zero by our choice of the cover $\{D(f_i)\}$, we are done. □

1.3.2. We have not used any noetherian hypothesis for proving Theorem 1.3.1.

1.3.3. Some history. Theorem 1.3.1 is sometimes called *Theorem B* or *Cartan's Theorem B*. This is because H. Cartan proved in the early 1950's that on a *Stein manifold*, or, more generally, on a *Stein analytic space*, *coherent analytic sheaves* are acyclic, and that theorem in several complex variables is called Cartan's Theorem B. Indeed Cartan labelled it Theorem B in [C]. Theorem A is the statement that coherent analytic sheaves on Stein spaces are generated by global sections. Affine varieties (and more generally affine schemes) are supposed to be analogous to Stein spaces. In 1956, Serre proved the converse to Cartan's Theorem B in the famous GAGA paper [S1] (he proved much much more in that paper than the converse). In 1957, Serre proved Theorem 1.3.1 for algebraic varieties over algebraically closed fields in [S2], and since that proof can be generalised *mutatis mutandis* to affine schemes, the result is generally attributed to Serre in the general case too, though it was formulated and proved in the general case by Grothendieck. There is also a converse to Theorem 1.3.1, again proved by Serre in [S2]. We prove the converse in Theorem 1.5.1 below.

1.4. **Quasi-affine schemes.** A scheme X is called *quasi-affine* if it is quasi-compact and isomorphic to an open subscheme of an affine scheme. Recall that an open subscheme of a scheme X is a ringed space of the form $(U,\, \mathscr{O}_X|_U)$ where U is a open subset of the topological space underlying X. As usual, we simply write U for this open subscheme, and note that $\mathscr{O}_U = \mathscr{O}_X|_U$.

1.4.1. For X a scheme and $f \in \Gamma(X,\, \mathscr{O}_X)$, we write X_f for the open subscheme $\{x \in X \mid f_x \notin \mathfrak{m}_x\}$, where, as aways, f_x is the germ of f at x and $\mathfrak{m}_x$ is the maximal ideal of $\mathscr{O}_{X,x}$. In other words X_f is what was denoted $D(f)$ in (1.1.1) of Lecture 9. We reserve the symbol $D(f)$ for affine schemes. In case X is affine, the notation $D(f)$ is preferred over X_f.

Proposition 1.4.2. *Let X be a scheme*

(a) *If X is quasi-compact and if the collection of open subschemes*

$$\mathfrak{U} = \{X_g \mid X_g \textit{ is affine}\}$$

is a cover of X, then the natural map $X \to \operatorname{Spec}(\Gamma(X,\, \mathscr{O}_X))$ of [Lecture 9, Corollary 1.1.4] is an open immersion with image $\bigcup_{X_g \in \mathfrak{U}} D(g)$. In particular X is quasi-affine.

(b) *If there exist $f_1, \dots, f_n \in \Gamma(X,\, \mathscr{O}_X)$ are such that X_{f_i} are affine, and $f_1, \dots, f_n$ generate the unit ideal in $\Gamma(X,\, \mathscr{O}_X)$, then X is affine.*

Proof. Let $A = \Gamma(X,\, \mathscr{O}_X)$. We ciam (b) follows from (a). To begin with, since the X_{f_i} are quasi-compact (being affine), X is quasi-compact, being a finite union of quasi-compact subsets. Since the f_i generate the unit ideal in A, the collection $\{X_{f_i}\}$ is an open cover of X. Indeed, we have $g_1, \dots, g_n \in A$ such that $\sum_i g_i f_i = 1$. If $x \in X$ is such that every $(f_i)_x$ is in $\mathfrak{m}_x$, then $1 \in \mathfrak{m}_x$ a contradiction. Hence part (a) applies and the natural map $X \to \operatorname{Spec} A$ is an open immersion with image $\bigcup_{X_f \in \mathfrak{U}} D(f)$. Since the f_i generate the unit ideal in A, we have $\bigcup_{X_f \in \mathfrak{U}} D(f) \supset \bigcup_{i=1}^{n} D(f_i) = \operatorname{Spec} A$, whence the image of X in $\operatorname{Spec} A$ is $\operatorname{Spec} A$ and we are done.

It remains to prove (a).

First note that if U and V are in $\mathfrak{U}$ then $U \cap V$ is affine. Indeed, if $U = X_g$ and $V = X_h$, then U and V are affine, say $U = \operatorname{Spec} R$ and $V = \operatorname{Spec} S$. The assertion follows from the relations $U \cap V = \operatorname{Spec}(R_{h|_U}) = \operatorname{Spec}(S_{g|_V})$. In particular, $U \cap V$ is quasi-compact.

Let

$$\Phi\colon X \to \operatorname{Spec} A$$

be the canonical map of [Lecture 9, Corollary 1.1.4]. Let $g \in A$. Since the image of g in $\Gamma(X_g,\, \mathscr{O}_X)$ is invertible, the universal property of localisation gives us a unique map of rings

$$\theta_g\colon A_g \to \Gamma(X_g,\, \mathscr{O}_X)$$

such that the composite $A \to A_g \to \Gamma(X_g,\, \mathscr{O}_X)$ is the restriction map. In simple terms, θ_g is the map $a/g^m \mapsto (g|_{X_g})^{-m}(a|_{X_g})$. We claim that the map θ_g is an isomorphism.

To that end let $U = \operatorname{Spec} R$ be an affine open subsecheme of X. Let g_R be the image of g in R under the natural restriction map $A \to R$, i.e. $g_R = g|_U$. Then clearly $U_{g_R} = U \cap X_g$, i.e. $D(g_R) = U \cap X_g$. Now suppose $a \in A$ is such that $a|_{X_g} = 0$. As before let a_R denote the image of a in R. Then $a_R/1 = 0$ in R_{g_R} for $a_R/1 = a|_{U\cap X_g}$. This means there exists an integer $\nu \geq 0$ such that $g_R^\nu a_R = 0$ in R.

Next let $b \in \Gamma(X_g,\, \mathscr{O}_X)$. Then $b|_{U\cap X_g}$ is an element of $\Gamma(U \cap X_g,\, \mathscr{O}_X) = \Gamma(D(g_R),\, \mathscr{O}_{\operatorname{Spec} R}))$. Since $\Gamma(D(g_R),\, \mathscr{O}_{\operatorname{Spec} R}) = R_{g_R}$, this means there is an integer $\mu \geq 0$ such that $b|_{U\cap X_g} = x/g_R^\mu$ for some $x \in R$. The sheaf theoretic way of putting it is that $g^\mu(b|_{U\cap X_g})$ extends to a section (namely x) of $\mathscr{O}_X$ over U.

Since X is quasi-compact and $\mathfrak{U}$ is a cover, we can find $U_1, \dots, U_d$ in $\mathfrak{U}$ such that $\bigcup_{i=1}^d U_i = X$. Moreover, from our observation earlier in this proof, $U_i \cap U_j$ is affine, whence quasi-compact.

Let $a \in A$ be as above, i.e. suppose $a|_{X_g} = 0$. For each $i = 1, \dots d$, we have, we have an integer $\nu_i \geq 0$ such that $(g|_{U_i})^{\nu_i}(a|_{U_i}) = 0$. Let ν be the maximum of the ν_i. Then $g^\nu a = 0$ since its restriction to each U_i is zero. This shows that the map θ_f is injective.

Now suppose $b \in \Gamma(X_g, \mathscr{O}_X)$. From what we have proved, there exists $\mu_1 \geq 0$ such that $g^{\mu_1} b|_{U_1\cap X_g}$ extends to a section of $\mathscr{O}_X$ over U_1. For $i = 1, \dots, d$, let $V_i = \bigcup_{j=1}^i U_j$. Suppose $i > 1$ and suppose we can find $\mu_{i-1} \geq 0$ such that $g^{\mu_{i-1}} b|_{V_{i-1}\cap X_g}$ extends to a section $s_{i-1} \in \Gamma(V_{i-1},\, \mathscr{O}_X)$. We may choose μ_{i-1} large enough that $g^{\mu_{i-1}} b|_{U_i \cap X_f}$ extends to a section $\sigma_i \in \Gamma(U_i,\, \mathscr{O}_X)$. On $W_i = V_{i-1} \cap U_i$ the section

$$\tau = s_{i-1}|_{W_i} - \sigma_i|_{W_i}$$

vanishes on $X_g \cap W_i = (W_i)_{g|_{W_i}}$. From what we have proved (with W_i playing the role of X, τ the role of a, and $g|_{W_i}$ the role of g), for each affine open subscheme U of W_i, we can find $\nu_U \geq 0$ such that $(g|_U)^{\nu_U}(\tau|_U) = 0$. Since $W_i = \bigcup_{k=1}^{i-1}(U_k \cap U_i)$ it is a finite union of quasi-compact sets and hence is quasi-compact and hence we can cover it by a finite number of affine open subschemes of the form U. Taking ν to be the maximum of the ν_U over this finite collection of U, we see that $(g|_{W_i})^\nu \tau = 0$. Unravelling the definition of τ we see that if $\mu_i := \nu + \mu_{i-1}$, then the section $g^{\mu_i} b|_{V_i \cap X_g}$ extends to a section $s_i \in \Gamma(V_i.\, \mathscr{O}_X)$. We have proved, by induction, that such $\mu_i \geq 0$ exist for $i = 1, \dots, d$. Setting $\mu = \mu_d$ we conclude that there exists $\mu \geq 0$ such that $(g^\mu|_{X_g})b$ extends to a section $s \in \Gamma(X,\, \mathscr{O}_X) = A$. This proves that $\theta_g(s/g^\mu) = b$ and hence $\theta_g\colon A_g \to \Gamma(X_g, \mathscr{O}_X)$ is surjective. Thus θ_g is an isomorphism as claimed.

Now suppose $X_g \in \mathfrak{U}$. Since X_g is affine, so that $X_g = \operatorname{Spec}\Gamma(X_g, \mathscr{O}_X)$, this means that the isomorphism θ_g gives us an isomorphism $X_g \xrightarrow{\sim} D(g)$ fitting into the commutative diagram

$$\begin{array}{ccc} X_g & \xrightarrow{\quad\sim\quad} & D(g) \\ \Vert & & \Vert \\ \operatorname{Spec}\Gamma(X_g, \mathscr{O}_X) & \xrightarrow[\text{via } \theta_g]{\sim} & \operatorname{Spec} A_g \end{array}$$

Now $X_g = \Phi^{-1}(D(g))$, where, recall, $\Phi\colon X \to \operatorname{Spec} A$ is the canonical map of [Lecture 9, Corollary 1.1.4]. We have therefore shown that for $X(g) \in \mathfrak{U}$ the maps

$$\Phi^{-1}(D(g)) \xrightarrow{\text{via } \Phi} D(g)$$

are isomorphisms. Since $\mathfrak{U}$ is a cover of X we are done. $\square$

1.4.3. The proof above can be understood more conceptually in the following way to bring out the role of the hypothesis of quasi-compactness on X. To begin with, suppose X is arbitrary (not necessarily quasi-compact). Let $\Phi\colon X \to \operatorname{Spec}\Gamma(X, \mathscr{O}_X)$ and $\Psi\colon X_g \to \operatorname{Spec}\Gamma(X_g, \mathscr{O}_{X_g})$ be the canonical maps of [Lecture 9, Corollary 1.1.4]. Then Φ and Ψ fit into a commutative diagram

$$\begin{array}{ccccc} X_g & = & \Phi^{-1}(D(g)) & \hookrightarrow & X \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \Phi|_{\Phi^{-1}(D(g))}} & & \downarrow{\scriptstyle \Phi} \\ \operatorname{Spec}\Gamma(X_g, \mathscr{O}_{X_g}) & \xrightarrow[\kappa]{} & D(g) & \hookrightarrow & \operatorname{Spec}\Gamma(X, \mathscr{O}_X) \end{array}$$

where κ is the map induced by the map $\theta_g\colon A_g \to \Gamma(X_g, \mathscr{O}_{X_g})$. It can also be interpreted as the unique map defined in Corollary 1.1.4 of Lecture 9, i.e. κ is the map f_Γ of *loc.cit.* for $f = \Phi|_{\Phi^{-1}(D(g))}$. If X_g is affine then Ψ is an isomorphism. If, on the other hand, X is quasi-compact, θ_g, and hence κ, is an isomorphism (whether X_g is affine or not). The latter fact is what takes up the bulk of the proof of part (a) of the proposition. From these observations, if X_g is affine *and* X is quasi-compact, the middle downward arrow is an isomorphism, since Ψ and κ are. If such X_g cover X, and X is quasi-compact, we get the conclusion of part (a) of the proposition.

1.4.4. Generalisation The proof that θ_g is an isomorphism when X is quasi-compact can easily be replicated to give the following: *Let X be quasi-compact and $g \in \Gamma(X, \mathscr{O}_X)$, then $\mathscr{F}$ a quasi-coherent sheaf on X. Then we have a natural isomorphism (induced by the universal property of localisation)*:

$$\Gamma(X, \mathscr{F})_g \xrightarrow{\sim} \Gamma(X_g, \mathscr{F}|_{X_g}). \tag{1.4.4.1}$$

We omit the proof since it is *mutatis mutandis* the proof we have given for $\mathscr{F} = \mathscr{O}_X$.

1.5. **The converse to Theorem B.** The following theorem is due to Serre [S2] (at least for varieties over algebraically closed fields).

Theorem 1.5.1. *Let X be a scheme. The following are equivalent*

(a) *X is affine.*

(b) X *is quasi-compact and* $\mathrm{H}^n(X, \mathscr{F}) = 0$ *for all* $n \geq 1$ *and all quasi-coherent* $\mathscr{O}_X$*-modules.*
(c) X *is quasi-compact and* $\mathrm{H}^1(X, \mathscr{I}) = 0$ *for all quasi-coherent ideal sheaves* $\mathscr{I}$ *of* $\mathscr{O}_X$*.*

Proof. Theorem 1.3.1 gives (a) $\Rightarrow$ (b). The implication (b) $\Rightarrow$ (c) is obvious. We now prove (c) $\Rightarrow$ (a).

Let U be an affine open subscheme of X. Let $Z = X \smallsetminus U$, and let $\mathscr{I}$ be the ideal sheaf of the reduced scheme structure on Z.

Let $x \in U$ be a closed point of X (and hence of U) and let $\mathfrak{m}_x$ be the quasi-coherent ideal sheaf of $\mathscr{O}_X$ for the reduced structure on $\{x\}$. Note that $\mathscr{O}_X/\mathfrak{m}_x$ is a skyscraper sheaf which is $\kappa(x)$, the residue field of $\mathscr{O}_{X,x}$, at x, and zero elsewhere. Let $\mathscr{I}'$ be the quasi-coherent ideal sheaf sheaf of $Z' = Z \cup \{x\}$ with Z' given the reduced scheme structure. Note that $\mathscr{I}' \subset \mathscr{I}$ and $\mathscr{I}/\mathscr{I}' = \mathscr{I}/\mathscr{I} \cap \mathfrak{m}_x = (\mathscr{I} + \mathfrak{m}_x)/\mathfrak{m}_x = \mathscr{O}_X/\mathfrak{m}_x$. This can also be seen by restricting $\mathscr{I}/\mathscr{I}'$ to $X \smallsetminus \{x\}$ where it is zero and to U where it is $\mathscr{O}_U/(\mathfrak{m}_x|_U)$, for $\mathscr{I}|_{X\smallsetminus\{x\}} = \mathscr{I}'|_{X\smallsetminus\{x\}}$, $\mathscr{I}|_U = \mathscr{O}_U$, and $\mathscr{I}'|_U = \mathfrak{m}_x|_U$. We thus have a short exact sequence

$$0 \longrightarrow \mathscr{I}' \longrightarrow \mathscr{I} \longrightarrow \mathscr{O}_X/\mathfrak{m}_x \longrightarrow 0.$$

Since $\mathrm{H}^1(X, \mathscr{I}') = 0$, this gives a surjective morphism $\Gamma(X, \mathscr{I}) \twoheadrightarrow \kappa(x)$. We therefore have an element $f \in \Gamma(X, \mathscr{I})$ which maps to $1 \in \kappa(x)$. Since $f \in \Gamma(X, \mathscr{I})$, it vanishes on Z. Thus $X_f \subset U$. On the other hand f does not vanish at x since its residue class in $\kappa(x)$ is 1. Thus X_f is an open neighbourhood of x. If $\bar{f} = f|_U$, then $U_{\bar{f}} = X_f$. Since U is affine, $U_{\bar{f}} = D(\bar{f})$ and hence is affine. Thus X_f is an affine open neighbourhood of x. Since x was an arbitrary closed point of X, we have a cover of X by open sets of the form X_f such that X_f are affine. We are using the fact that if R is a ring, J an R-ideal, then $V(J)$ must contain a maximal ideal, whence any open set of $\operatorname{Spec} R$ containing all the maximal ideals of R must be all of $\operatorname{Spec} R$.

Since X is quasi-compact, the above discussion shows that we have $f_1, \ldots, f_n$ in $\Gamma(X, \mathscr{O}_X)$ such that the $X(f_i)$ are affine and cover X. By part (b) of Proposition 1.4.2 we are done if we show that $f_1, \ldots, f_n$ generate the unit ideal in $\Gamma(X, \mathscr{O}_X)$. We proceed to do this now. We claim the sequence

$$(\#) \qquad 0 \longrightarrow \mathscr{F} \longrightarrow \bigoplus_{i=1}^{n} \mathscr{O}_X \xrightarrow{(f_1,\ldots,f_n)} \mathscr{O}_X \longrightarrow 0$$

is exact where $\mathscr{F} \to \bigoplus_{i=1}^n \mathscr{O}_X$ is the defined as the kernel of the map labelled $(f_1, \ldots, f_n)$. The map $(f_1, \ldots, f_n)$ on any affine open $\operatorname{Spec} R$ is the map $\bigoplus_{i=1}^n R \to R$ given by the row matrix $(f, \ldots, f_n)$, i.e. $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n f_i x_i$. Since $\{X_{f_i}\}_{i=1}^n$ is a cover of X, in a neighbourhood of each point at least one f_i is invertible, which shows that the map labelled $(f_1, \ldots, f_n)$ is an epimorphism. Thus (#) is an exact sequence.

Let $\mathscr{V}_i = \bigoplus_{j=1}^i \mathscr{O}_X$ for $i = 1, \ldots, n$ and $\mathscr{V}_0 = 0$. Set $\mathscr{F}_i = \mathscr{F} \cap \mathscr{V}_i$ for $i = 0, \ldots, n$. We then have a filtration of $\mathscr{F}$:

$$0 = \mathscr{F}_0 \subset \mathscr{F}_1 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_n = \mathscr{F}.$$

The successive quotients $\mathscr{F}_i/\mathscr{F}_{i-1} = (\mathscr{F}_i + \mathscr{V}_{i-1})/\mathscr{V}_{i-1}$ are submodules of $\mathscr{V}_i/\mathscr{V}_{i-1} = \mathscr{O}_X$ for $i = 1, \ldots, n$ and thus are quasi-coherent ideal sheaves. In particular $\mathrm{H}^1(X, \mathscr{F}_i/\mathscr{F}_{i-1}) = 0$ for $i = 1, \ldots, n$. Thus, if for some $i \in \{1, \ldots, n\}$ we have

$\mathrm{H}^1(X, \mathscr{F}_{i-1}) = 0$, then $\mathrm{H}^1(X, \mathscr{F}_i) = 0$ via the long exact sequence associated to the exact sequence $0 \to \mathscr{F}_{i-1} \to \mathscr{F}_i \to \mathscr{F}_i/\mathscr{F}_{i-1} \to 0$. Since $\mathrm{H}^1(X, \mathscr{F}_0) = 0$, by induction we see that $\mathrm{H}^1(X, \mathscr{F}) = 0$. Applying this to the long exact sequence in cohomology associated to (#), we see that the map $\bigoplus_{i=1}^n \Gamma(X, \mathscr{O}_X) \xrightarrow{(f_1,\ldots,f_n)} \Gamma(X, \mathscr{O}_X)$ is surjective. This means there exist $g_1, \ldots, g_n \in \Gamma(X, \mathscr{O}_X)$ such that $\sum_{i=1}^n g_i f_i = 1$. This is what we were supposed to prove. $\square$

1.5.2. Once again note that we have not used any noetherian hypothesis anywhere.

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