LECTURES 10 AND 11

Dates of Lectures: Oct 26 and 28, 2021

The symbol $\mathcal{A}b$ will denote the category of abelian groups and $\mathbb{S}\mathrm{ch}$ the category of schemes. If X is a topological space, $\mathcal{P}sh_X$ and $\mathcal{S}h_X$ denote the category of presheaves and the category of sheaves respectively on X. By a ring we mean a commutative ring with identity. For a ring A, Mod_A denotes the category of A-modules. For a sheaf of rings $\mathscr{A}$ on a topological space, $\mathrm{Mod}_{\mathscr{A}}$ will denote the category of $\mathscr{A}$-modules.

The symbol ☡ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

1. More on derived functors

1.1. **Acyclic objects.** Let $\mathcal{A}$, $\mathcal{B}$ be abelian categories, $\mathcal{A}$ with enough injectives, and $T\colon \mathcal{A} \to \mathcal{B}$ a left exact functor. Note that

$$\mathrm{R}^0T = T \tag{1.1.1}$$

since T is left exact. Indeed, if $A \in \mathcal{A}$ and $A \to E^\bullet$ is a classical injective resolution, then $\mathrm{R}^0T(A) = \mathrm{H}^0(0 \to T(E^0) \to T(E^1) \to \dots) = T(A)$ by the left exactness of T. One can check that this identification is functorial.

Definition 1.1.2. An object A of $\mathcal{A}$ is said to be T-*acyclic* if $\mathrm{R}^iT(A) = 0$ for $i \geq 1$. A complex $C^\bullet$ consisting of T-acyclic objects is called a T-acyclic complex.

1.1.3. If E is an injective object of $\mathcal{A}$ then it is T-acyclic, for $\mathbf{1}_E\colon E \to E$ is an injective resolution of E.

Lemma 1.1.4. *Let $C^\bullet$ be a bounded below exact complex of T-acyclic objects in $\mathcal{A}$. Then $T(C^\bullet)$ is an exact complex.*

Proof. Without loss of generality we assume that $C^n = 0$ for negative n. Let $Z^i = Z^i(C^\bullet)$ and $B^i = B^i(C^\bullet)$ have their usual meaning as the i^{th} cocycle object of $C^\bullet$ and the i^{th} coboundary object of $C^\bullet$ respectively. Since $C^\bullet$ is exact, $Z^i = B^i$ for all i. We therefore have exact sequences

$$0 \longrightarrow Z^i \longrightarrow C^i \to Z^{i+1} \longrightarrow 0 \qquad i \longrightarrow 0. \tag{$(*)_i$}$$

If Z^i is T-acyclic for some i then applying the long exact sequence of derived functors to $(*)$ we see that

$$0 \longrightarrow T(Z^i) \longrightarrow T(C^i) \longrightarrow T(Z^{i+1}) \longrightarrow 0 \tag{$(\dagger)_i$}$$

and that $\mathrm{R}^nT(Z^{i+1}) = 0, n \geq 1$. In other words Z^{i+1} is also T-acyclic. Now $Z^0 = 0$ is T-acyclic, and hence by induction Z^i is T-acyclic for all i. Hence the sequence $(\dagger)_i$ is exact for all i. By the left exactness of T we know that $Z^i(T(C^\bullet)) = T(Z^i)$ for all i. From this and $(\dagger)_i$ we see that $B^{i+1}(T(C^\bullet)) = Z^{i+1}(T(C^\bullet))$ for all integers i. It follows that $T(C^\bullet)$ is exact. □

Proposition 1.1.5. *Let $\varphi\colon X^\bullet \to Y^\bullet$ be a quasi-isomorphism between bounded below T-acyclic complexes in $\mathcal{A}$. Then $T(\varphi)\colon T(C^\bullet) \to T(D^\bullet)$ is also a quasi-isomorphism.*

Proof. Since φ is a quasi-isomorphism, its mapping cone $C^\bullet_\varphi$ is exact. Clearly $C^\bullet_\varphi$ is bounded below, and since $C^n_\varphi = X^{n+1} \oplus Y^n$ and T is additive (so that so are R^iT for all i) we see that C^n_φ is T-acyclic for each n. Hence $C^\bullet_\varphi$ is a bounded below exact T-acyclic complex. By Lemma 1.1.4 we deduce that $T(C^\bullet_\varphi)$ is exact. Now clearly $T(C^\bullet_\varphi) = C^\bullet_{T(\varphi)}$. Hence $T(\varphi)$ is a quasi-isomorphism. □

The following theorem is sometimes called the abstract de Rham theorem.

Theorem 1.1.6. *Let $A \in \mathcal{A}$ and $A \to X^\bullet$ a resolution of A by a bounded below T-acyclic complex. Then there for each i there is a canonical isomorphism*

$$\mathrm{H}^i(T(X^\bullet)) \xrightarrow{\sim} \mathrm{R}^iT(A).$$

Proof. Let $A \to E^\bullet$ be a bounded below injective resolution of A. In the homotopy category $\mathbf{K}(\mathcal{A})$ we have a unique map $\varphi\colon X^\bullet \to E^\bullet$ such that the composite $A \to X^\bullet \xrightarrow{\varphi} E^\bullet$ is this injective resolution. Taking cohomologies of the complexes we see that φ is a quasi-isomorphism. From Proposition 1.1.5 we see that $T(\varphi)$ is a quasi-isomorphism. The theorem follows. □

1.1.7. One can also prove that $\mathrm{H}^i(T(X^\bullet))$ computes $\mathrm{R}^iT(A)$ be breaking the augumented complex $0 \to A \to X^\bullet$ into short exact sequences and using an induction argument. The two methods of getting the isomorphism are not exactly the same. They differ by a factor of $(-1)^{i(i-1)/2}$. See if you can work this out. In this course we will use the isomorphism given by the proof of the theorem. It is the most natural one.

2. Cohomology of sheaves

Fix a topological space X. The functor $\Gamma = \Gamma(X, -)$ is the global sections functor as in Homework 3. Recall from Theorem 3.2.6 of Lecture 9 that $\mathcal{Sh}_X$ is an abelian category. In fact the same proof shows that $\mathrm{Mod}_{\mathscr{A}}$ is an abelian category for any sheaf of rings $\mathscr{A}$ on X.

The symbol $\mathbf{Z}_X$ will denote the sheaf of locally constant integer valued functions on X. An equivalent way of defining $\mathbf{Z}_X$ is this: If $\mathbf{Z}_{pr}$ is the constant presheaf $U \mapsto \mathbf{Z}$, with restriction maps being the identity map, then $\mathbf{Z}_X = \mathbf{Z}^+_{pr}$, the sheafification of $\mathbf{Z}_{pr}$.

2.1. The sheaf $G(\mathscr{F})$ associated to a sheaf $\mathscr{F}$.

Let $\mathscr{A}$ sheaf of rings on X and $\mathscr{F}$ an $\mathscr{A}$-module. Define $G(\mathscr{F}) = G^0(\mathscr{F})$ to be the sheaf

$$U \mapsto \prod_{x \in U} \mathscr{F}_x, \tag{2.1.1}$$

with obvious restriction maps. This is *a priori* a presheaf but it is clearly a sheaf. One way of thinking about $G(\mathscr{F})$ is that it is the sheaf of (possibly discontinuous) set theoretic sections of $\pi\colon \mathscr{E}(\mathscr{F}) \to X$, where $\mathscr{E}(\mathscr{F})$ is the étale space of $\mathscr{F}$. In other words, for an open set U, $\mathscr{E}(\mathscr{F})(U)$ is the $\mathscr{A}(U)$ module of set-theoretic functions $\sigma\colon U \to \mathscr{E}(\mathscr{F})$ (with no requirement of continuity) such that $\pi \circ \sigma = \mathbf{1}_U$. The sheaf $G(\mathscr{F})$ is often called *the sheaf of discontinuous sections* of $\mathscr{F}$. Or sometimes the *Godement sheaf* of F after R. Godement who first defined them. Note that

$$G\colon \mathrm{Mod}_{\mathscr{A}} \longrightarrow \mathrm{Mod}_{\mathscr{A}} \tag{2.1.2}$$

is a functor, since a map $\mathscr{F} \to \mathscr{G}$ in $\mathrm{Mod}_{\mathscr{A}}$ gives us a map $\mathscr{F}_x \to \mathscr{G}_x$ in $\mathrm{Mod}_{\mathscr{A}_x}$ for every $x \in X$, which in turn gives us a map $G(\mathscr{F}) \to G(\mathscr{G})$ in $\mathrm{Mod}_{\mathscr{A}}$. We have a functorial inclusion of $\mathscr{A}$-modules

$$\mathscr{F} \hookrightarrow G(\mathscr{F}). \tag{2.1.3}$$

The inclusion is the obvious inclusion of continuous sections of $\pi\colon \mathscr{E}(\mathscr{F}) \to X$ into set theoretic sections. It is functorial in the sense that it gives a natural transformation $\mathbf{1} \to G$ of functors from $\mathscr{A}$ to $\mathscr{A}$.

2.2. $\mathrm{Mod}_{\mathscr{O}_X}$ **has enough injectives.** Let $\mathscr{O}_X$ be a sheaf of rings on X. We will show that the abelian category $\mathrm{Mod}_{\mathscr{O}_X}$ has enough injectives. We point out that with $\mathbf{Z}_X$ the sheaf defined above, $\mathcal{S}h_X = \mathrm{Mod}_{\mathbf{Z}_X}$. Thus this result will show that $\mathcal{S}h_X$ has enough injectives.

Let $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_X}$. We know from §1.1.1 of Lecture 8 that Mod_A has enough injectives for every ring A. in particular $\mathrm{Mod}_{\mathscr{O}_{X,x}}$ has enough injectives for every $x \in X$. Choose an embedding $\mathscr{F}_x \hookrightarrow E_x$ of $\mathscr{F}_x$ into an injective $\mathscr{O}_{X,x}$-module. Define a sheaf $\mathscr{E}$ by the rule (with obvious restrictions)

$$U \mapsto \prod_{x \in U} E_x.$$

Note that $G(\mathscr{F}) \hookrightarrow \mathscr{E}$ whence we have an inclusion $\mathscr{F} \hookrightarrow \mathscr{E}$ via (2.1.3).

We claim that $\mathscr{E}$ is an injective object in $\mathrm{Mod}_{\mathscr{A}}$. For each $x \in X$, let i_x be the inclusion $\{x\} \subset X$. Regard E_x as a sheaf on $\{x\}$. Then

$$\mathscr{E} = \prod_{x \in X} (i_x)_* E_x \tag{2.2.1}$$

Now for any $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_X}$, we have $\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{F}, (i_x)_*E_x) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{O}_{X,x}}(\mathscr{F}_x, \mathscr{E}_x)$. Since talking stalks is an exact functor and since $\mathrm{Hom}_{\mathscr{O}_{X,x}}(-, E_x)$ is an exact functor, it follows that $\mathrm{Hom}_{\mathscr{O}_X}(-, (i_x)_*E_x)$ is exact. In other words $(i_x)_*E_x$ is an injective object in $\mathrm{Mod}_{\mathscr{A}}$. Since the arbitrary product of injective sheaves is injective (very easy to prove) we conclude, by (2.2.1), that $\mathscr{E}$ is an injective object of $\mathrm{Mod}_{\mathscr{A}}$.

2.3. **The functor** $\mathrm{H}^i(X, -)$**.** For $i \in \mathbf{Z}$,

$$\mathrm{H}^i(X, -)\colon \mathcal{S}h_X \to \mathcal{A}b$$

is defined to be the i^{th} derived functor of $\Gamma(X, -)\colon \mathcal{S}h_X \to \mathcal{A}b$. In other words, if $\mathscr{F} \in \mathcal{S}h_X$, then

$$\mathrm{H}^i(X, \mathscr{F}) := \mathrm{H}^i(\Gamma(X, \mathscr{E}^\bullet)) \tag{2.3.1}$$

where $\mathscr{F} \to \mathscr{E}^\bullet$ is a bounded below injective resolution of $\mathscr{F}$. The group $\mathrm{H}^i(X, \mathscr{F})$ is called the i^{th} *cohomology group of* $\mathscr{F}$ or sometimes the i^{th} *cohomology of* X *with coefficients in* $\mathscr{F}$.

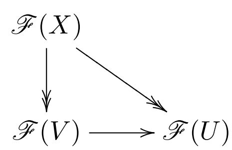
3. **Flasque sheaves**

From Theorem 1.1.6 we see that one way of computing the cohomology groups of $\mathscr{F} \in \mathcal{S}h_X$ is to resolve $\mathscr{F}$ by $\Gamma(X, -)$-acyclic objects. In sheaf theory, these are simply called *acyclic sheaves*. The standard acyclic sheaves one uses are what are called *flasque sheaves*.[1]

3.1. A flasque sheaf is one for which sections over an open set can be extended to a global sections. The formal definition is below.

Definition 3.1.1. A sheaf $\mathscr{F}$ on X is called *flasque* if for every open set U in X the map $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective.

If $\mathscr{F}$ is flasque and V is an open subset of X, then $\mathscr{F}|_V$ is also flasque. To see this, suppose U is an open subset of V. The following diagram commutes (with all arrows being restriction maps):



Since the southeast pointing arrow is an epimorphism (for $\mathscr{F}$ is flasque), the horizontal arrow is also an epimorphism. Indeed given ξ in $\mathscr{F}(U)$, there exists $\xi^* \in \mathscr{F}(X)$ such that $\xi^*|_U = \xi$. Then $\zeta := \xi^*|_V$ is an element of $\mathscr{F}(V)$ such that $\zeta|_U = \xi$, proving that $\mathscr{F}|_V$ is flasque.

It is clear that the Godement sheaf $G(\mathscr{F})$ is a flasque sheaf for any sheaf $\mathscr{F}$. In fact if $\mathscr{O}_X$ is a sheaf of rings on X and $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_X}$, then by (2.1.3), we see that $\mathscr{F}$ can be embedded as an $\mathscr{O}_X$-submodule into a flasque $\mathscr{O}_X$-module. In other words $\mathrm{Mod}_{\mathscr{O}_X}$ *has enough flasques*.

3.1.2. The Godement resolution. As before let $G^0(\mathscr{F}) = G(\mathscr{F})$ be the Godement sheaf defined in §2.1. Suppose $G^i(\mathscr{F})$ and $d^{i-1}\colon G^{i-1}(\mathscr{F}) \to G^i(\mathscr{F})$ have been defined for $i = 0, \dots, n$ in such a way that

$$0 \longrightarrow \mathscr{F} \xrightarrow{(2.1.3)} G^0(\mathscr{F}) \longrightarrow \dots \longrightarrow G^{n-1}(\mathscr{F}) \longrightarrow G^n(\mathscr{F})$$

is exact, and $G^0(\mathscr{F}), \dots, G^n(\mathscr{F})$ are flasque. Define $G^{n+1}(\mathscr{F}) = G(\mathrm{coker}\,(d^{n-1}))$ and $d^n\colon G^n(\mathscr{F}) \to G^{n+1}(\mathscr{F})$ as the composite

$$G^n(\mathscr{F}) \twoheadrightarrow \mathrm{coker}\,(d^{n-1}) \hookrightarrow G(\mathrm{coker}\,(d^{n-1})) = G^{n+1}(\mathscr{F}).$$

This defines $G^i(\mathscr{F})$ recursively for $i \geq 0$. Moroever G^i is a functor. We thus have a functorial flasque resolution

$$0 \xrightarrow{(2.1.3)} \mathscr{F} \longrightarrow G^0(\mathscr{F}) \longrightarrow G^1(\mathscr{F}) \longrightarrow \dots \longrightarrow G^n(\mathscr{F}) \longrightarrow \dots. \tag{3.1.2.1}$$

This is called *the canonical flasque resolution* of $\mathscr{F}$ or *the Godement resolution* of $\mathscr{F}$. It is functorial in $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_X}$. One can write an "exact sequence of functors"

$$0 \to \mathbf{1} \to G^0 \to G^1 \to \dots \to G^n \to \dots$$

which can be regarded as a resolution of the identity functor on $\mathrm{Mod}_{\mathscr{O}_X}$ by the complex $G^\bullet$ of functors which take values in flasque $\mathscr{O}_X$-modules.

[1]Also called *flabby sheaves*.

The crucial result which one needs to show that flasque sheaves are acyclic is the following.

Proposition 3.1.3. *Let*

$$0 \longrightarrow \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{H} \longrightarrow 0$$

be a short exact sequence of sheaves.

(a) *If $\mathscr{F}$ is flasque then the sequence*

$$0 \longrightarrow \Gamma(X, \mathscr{F}) \xrightarrow{\alpha(X)} \Gamma(X, \mathscr{G}) \xrightarrow{\beta(X)} \Gamma(X, \mathscr{H}) \longrightarrow 0$$

is exact.

(b) *If $\mathscr{F}$ and $\mathscr{G}$ are flasque, then $\mathscr{H}$ is flasque.*

Proof. Let us first prove (a). Suppose $\sigma \in \Gamma(X, \mathscr{H})$. Let Λ be the set of pairs (s, V) with V an open set of X and s a section of $\mathscr{G}$ over V such that $\beta(V)(s) = \sigma|_V$. There is a natural partial order $\prec$ on Λ given by $(s, V) \prec (t, W)$ if $V \subset W$ and $t|_V = s$. It is clear that every totally ordered chain in Λ is bounded above since $\mathscr{G}$ is a sheaf. Hence Zorn's Lemma applies and we have a maximal element (V^*, s^*) in Λ. If $V^* = X$ we are done. Suppose this is not so. Pick a point $x \in X \smallsetminus V^*$. The map $\mathscr{G}_x \to \mathscr{H}_x$ is an epimorphism (since $\mathscr{G} \to \mathscr{H}$ is) and hence the germ σ_x of σ at x has a pre-image s_x in $\mathscr{G}_x$. It follows that there is an open neighbourhood U of x and an element $s \in \Gamma(U, \mathscr{G})$ such that $\beta(U)(s) = \sigma|_U$. Let $W = V^* \cap U$. Then $\beta(W)(s^*|_W) = \sigma|_W = \beta(W)(s|_W)$. Thus $\beta(W)(s^*|_W - s|_W) = 0$. Since $\ker(\beta(W)) = \Gamma(W, \mathscr{F})$, we have an element $\tau \in \Gamma(W, \mathscr{F})$ such that $\alpha(W)(\tau) = s^*|_W - s|_W$. Now $\mathscr{F}$ is flasque. Hence there exists $\theta \in \Gamma(U, \mathscr{F})$ such that $\theta|_U = \tau$. Let $t \in \Gamma(U, \mathscr{G})$ be the image of θ under the inclusion $\alpha(U)$. Set $s' = s + t \in \Gamma(U, \mathscr{G})$. Then $s'|_W = s^*|_W$, whence we get a section $s^{**} \in \Gamma(V^* \cup U, \mathscr{G})$ such that $s^{**}|_{V^*} = s^*$. Moreover, clearly $\beta(V^* \cup U)(s^{**}) = \sigma|_{V^* \cup U}$. The pair $(s^{**}, V^* \cup U)$ lies in Λ and violates the maximality of (s^*, V^*). This is a contradiction which proves that $V^* = X$ and we are done for part (a).

We now prove (b). Let U be an open set in X and σ and element of $\Gamma(U, \mathscr{H})$. By part(a) σ has a pre-image $s \in \Gamma(U, \mathscr{G})$, since $\mathscr{F}|_U$ is flasque. Since $\mathscr{G}$ is flasque, s can be extended to a section $t \in \Gamma(X, \mathscr{G})$. Let $\tau = \beta(X)(t)$. Then $\tau \in \Gamma(X, \mathscr{H})$ and $\tau|_U = \sigma$. □

Lemma 3.1.4. *Let $\mathscr{G}^\bullet$ be a bounded below exact sequence of flasque sheaves on X. Then $\Gamma(X, \mathscr{G}^\bullet)$ is an exact complex.*

Proof. Without loss of generality we assume that $\mathscr{G}^n = 0$ for negative n. For $i \in \mathbf{Z}$, let $\mathscr{Z}^i = \ker(\mathscr{G}^i \to \mathscr{G}^{i+1})$ and $\mathscr{B}^i = \mathbf{im}(\mathscr{G}^{i-1} \to \mathscr{G}^i)$. Since $\mathscr{G}^\bullet$ is exact, we have $\mathscr{Z}^i = \mathscr{B}^i$ for all i. For each $i \geq 0$ (in fact for $i \in \mathbf{Z}$) we therefore have an exact sequence

$$(*)_i \qquad 0 \longrightarrow \mathscr{Z}^i \longrightarrow \mathscr{G}^i \longrightarrow \mathscr{Z}^{i+1} \longrightarrow 0.$$

Now $\mathscr{Z}^0 = 0$ is flasque. Suppose $\mathscr{Z}^i$ is flasque for some $i \geq 0$. Then by (b) of Proposition 3.1.3 and $(*)_i$, we conclude that $\mathscr{Z}^{i+1}$ is also flasque, since $\mathscr{G}^i$ is. By induction it follows that $\mathscr{Z}^i$ is flasque for all $i \in \mathbf{Z}$ (for negative i this is obvious, since $\mathscr{Z}^i = \mathscr{G}^i = 0$). By part (a) of Proposition 3.1.3 it follows that

$$(\dagger)_i \qquad 0 \longrightarrow \Gamma(X, \mathscr{Z}^i) \longrightarrow \Gamma(X, \mathscr{G}^i) \longrightarrow \Gamma(X, \mathscr{Z}^{i+1}) \longrightarrow 0$$

is exact for every i. Since $\Gamma(X,\,-)$ is left exact,

$$\Gamma(X,\,\mathscr{Z}^i) = \ker\left(\Gamma(X,\,\mathscr{G}^i) \to \Gamma(X,\,\mathscr{G}^{i+1})\right) \qquad (i \in \mathbf{Z}).$$

Applying this to $(\dagger)_i$ we see that $\Gamma(X,\,\mathscr{Z}^{i+1}) = \mathbf{im}\left(\Gamma(X,\,\mathscr{G}^i) \to \Gamma(X,\,\mathscr{G}^{i+1})\right)$ for all $i \in \mathbf{Z}$. Replacing i by $i-1$ in the last observation we see that

$$\mathbf{im}\left(\Gamma(X,\,\mathscr{G}^{i-1}) \to \Gamma(X,\,\mathscr{G}^i)\right) = \ker\left(\Gamma(X,\,\mathscr{G}^i) \to \Gamma(X,\,\mathscr{G}^{i+1})\right)$$

for all $i \in \mathbf{Z}$. This proves the lemma. $\square$

4. Higher cohomologies of flasque sheaves

The main result is Theorem 4.2.3 below which states that flasques are acyclic.

4.1. Extension of a sheaf by zero. Let U be an open subset of X and $i\colon U \to X$ the inclusion map. Set

$$i_!\mathscr{F}(V) = \begin{cases} 0 & \text{if } V \not\subset U \\ \mathscr{F}(V) & \text{if } V \subset U \end{cases} \tag{4.1.1}$$

It is clear that $i_!\mathscr{F}$ is a sheaf with restrictions being the obvious ones, and $(i_!\mathscr{F})|_U = \mathscr{F}|_U$. In fact we have a canonical inclusion

$$i_!\mathscr{F} \subset \mathscr{F}. \tag{4.1.2}$$

Note that $i_!$ is exact (see how it behaves on stalks).

The following formula is obvious for $\mathscr{O}_X$-modules $\mathscr{F}$ and $\mathscr{G}$.

$$\mathrm{Hom}_{\mathscr{O}_X}(i_!\mathscr{F},\,\mathscr{G}) = \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{F}|_U,\,\mathscr{G}|_U). \tag{4.1.3}$$

4.2. Flasque sheaves are acyclic. Recall that if A is a ring and M an A-module then $\mathrm{Hom}_A(A,\,M)$ is canonically identified with M via "evaluation at 1", i.e. via the map $\phi \mapsto \phi(1)$. Moreover this identification is functorial in $M \in \mathrm{Mod}_A$. Along these lines, if $\mathscr{O}_X$ is a sheaf of rings on X and $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_X}$, then we have a canonical functorial identification

$$\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X,\,\mathscr{F}) = \Gamma(X,\,F). \tag{4.2.1}$$

In (4.2.1), $\varphi \in \mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X,\,\mathscr{F})$ gets identified via "evaluation at 1" with $\varphi(X)(1) \in \Gamma(X,\,\mathscr{F})$. Conversely, given $\sigma \in \Gamma(X,\,\mathscr{F})$, the map φ from $\mathscr{O}_X$ to $\mathscr{F}$ is $\varphi(U)(x) = x(\sigma|_U)$, $x \in \mathscr{O}_X(U)$.

With these preliminaries we have the following theorem:

Theorem 4.2.2. *Let $\mathscr{O}_X$ be a sheaf of rings on X and $\mathscr{E}$ an injective object in $\mathrm{Mod}_{\mathscr{O}_X}$, i.e. $\mathscr{E}$ is an injective $\mathscr{O}_X$-module. Then $\mathscr{E}$ is flasque.*

Proof. Let U be an open subset of X. Consider the inclusion $i_!\mathscr{O}_X \subset \mathscr{O}_X$ as in (4.1.2). We have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathscr{O}_X}(\mathscr{O}_X,\,\mathscr{E}) & \xrightarrow{\text{via (4.1.2)}} & \mathrm{Hom}_{\mathscr{O}_X}(i_!\mathscr{O}_X,\,\mathscr{E}) \\
\| & & \| \,(4.1.3) \\
(4.2.1)\, \| & & \mathrm{Hom}_{\mathscr{O}_U}(\mathscr{O}_U,\,\mathscr{E}|_U) \\
\| & & \| \,(4.2.1) \\
\Gamma(X,\,\mathscr{E}) & \xrightarrow{\text{restriction}} & \Gamma(U,\,\mathscr{E})
\end{array}$$

Since $\mathscr{E}$ is an injective $\mathscr{O}_X$-module, the functor $\mathrm{Hom}_{\mathscr{O}_X}(-, \mathscr{E})$ is exact and hence converts the inclusion $i_!\mathscr{O}_X \subset \mathscr{O}_X$ into a surjection, whence the horozontal arrow on the top of the diagram is a surjection. It follows that the horizontal arrow at the bottom is also a surjection. □

Here is the main result about flasque sheaves:

Theorem 4.2.3. *Let $\mathscr{F}$ be a flasque sheaf. Then $\mathscr{F}$ is acyclic, i.e. $\mathrm{H}^i(X, \mathscr{F}) = 0$ for $i \geq 1$.*

Proof. Let $\mathscr{F} \to \mathscr{E}^\bullet$ be a classical injective resolution. By Theorem 4.2.2 and Lemma 3.1.4 we are done. □

5. **A basic result on the first cohomology**

5.1. **Direct images again.** Let X be a topological space and U an open subset of X and $i\colon U \subset X$ the inclusion map. For $\mathscr{F} \in \mathcal{S}\!h_X$ we define

$$ {}_U\mathscr{F} = i_*(\mathscr{F}|_U). \tag{5.1.1}$$

It is clear that ${}_U(-)$ is a functor.

If $\mathscr{O}_X$ is a sheaf of rings on X, we have (as is easily checked) ${}_U\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_X}$ whenever $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_X}$. We point out that for an open subset V of X

$$ {}_U\mathscr{F}(V) = \mathscr{F}(U \cap V). \tag{5.1.2}$$

We therefore have a natural map

$$ \mathscr{F} \longrightarrow {}_U\mathscr{F} \tag{5.1.3}$$

which on an open set V is the restriction $\mathscr{F}(V) \to \mathscr{F}(U \cap V)$. Note that if $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_X}$, then (5.1.3) is a map of $\mathscr{O}_X$-modules.

Here is a general result about direct images of sheaves.

Proposition 5.1.4. *Let $f\colon X \to Y$ be a continuous map between topological space and $\mathscr{F}$ a flasque sheaf on X. Then $f_*\mathscr{F}$ is flasque. In particular ${}_U\mathscr{F}$ is flasque for every open set U in X*

Proof. Let V be an open subset of Y and $U = f^{-1}(V)$. Now $\mathscr{F}(X) \to \mathscr{F}(U)$ is surjective. On the other hand $\mathscr{F}(X) = f_*\mathscr{F}(Y)$ and $\mathscr{F}(U) = f_*\mathscr{F}(V)$. It follows that $f_*\mathscr{F}$ is flaque.

The second part follows from the first part and the fact that $\mathscr{F}|_U$ is flasque. □

The following lemma is the first step towards proving that quasi-coherent sheaves are acyclic on affine schemes (see [K]).

Lemma 5.1.5. *Let $\mathcal{B}$ be a base for the topology on X. Let $\mathscr{F} \in \mathcal{S}\!h_X$ and let $\alpha \in \mathrm{H}^1(X, \mathscr{F})$. Then there exists an open cover $\mathfrak{V}$ of X, with $\mathfrak{V} \subset \mathcal{B}$, such that the image of α in $\mathrm{H}^1(X, {}_V\mathscr{F})$, under the map $\mathrm{H}^1(X, \mathscr{F}) \to \mathrm{H}^1(X, {}_V\mathscr{F})$ induced by (5.1.3), is zero for every $V \in \mathfrak{V}$.*

Proof. Since $\mathcal{S}\!h_X$ has enough flasques, we have a monomorphism $j\colon \mathscr{F} \hookrightarrow \mathscr{G}$ with $\mathscr{G}$ flasque. Let $\pi\colon \mathscr{G} \to \mathscr{H}$ be the cokernel of j. Then we have a short exact sequence of sheaves

$$ 0 \longrightarrow \mathscr{F} \xrightarrow{i} \mathscr{G} \xrightarrow{\pi} \mathscr{H} \longrightarrow 0 \tag{$*$}$$

Since $\mathscr{G}$ is flasque, and hence acyclic, we get an exact sequence

$$ 0 \longrightarrow \Gamma(X, \mathscr{F}) \xrightarrow{i(X)} \Gamma(X, \mathscr{G}) \xrightarrow{\pi(X)} \Gamma(X, \mathscr{H}) \xrightarrow{\delta} \mathrm{H}^1(X, \mathscr{F}) \longrightarrow 0 \tag{$\dagger$}$$

with δ the connecting map. It follows that there exists $\beta \in \Gamma(X, \mathscr{H})$ such that $\delta(\beta) = \alpha$.

Let $x \in X$ be a point. The map $\pi_x \colon \mathscr{G}_x \to \mathscr{H}_x$ is surjective, whence we can find $\gamma_x \in \mathscr{G}_x$ such that $\pi_x(\gamma_x) = \beta_x$. It follows we can find an open neighbourhood V of x and $\gamma_V \in \mathscr{G}(V)$ such that $\pi(V)(\gamma_V) = \beta|_V$. Moreover, since $\mathcal{B}$ is a base for the topology on X, we may choose $V \in \mathcal{B}$. As we vary x in X we get a collection of V in $\mathcal{B}$ and this gives us a cover $\mathfrak{V}$ of X with $\mathfrak{V} \subset \mathcal{B}$ as well a collection of local sections $\{\gamma_V \mid V \in \mathfrak{V}\}$ of $\mathscr{G}$ as above.

Fix $V \in \mathfrak{V}$. Consider the commutative diagram

$$\begin{array}{ccc} \mathscr{G} & \xrightarrow{\quad\pi\quad} & \mathscr{H} \\ {\scriptstyle (5.1.3)}\big\downarrow & & \big\downarrow{\scriptstyle (5.1.3)} \\ {}_V\mathscr{G} & \xrightarrow[\text{via } \pi]{\qquad} & {}_V\mathscr{H} \end{array} \tag{5.1.5.1}$$

The natural map $\mathscr{G} \to {}_V\mathscr{G}$ is an epimorphism since $\mathscr{G}$ is flasque. In greater detail, this map is the one described in (5.1.3) and over an open set W is the natural restriction $\mathscr{G}(W) \to \mathscr{G}(V \cap W)$ which is surjective since $\mathscr{G}$ is flasque. Thus in the commutative diagram (5.1.5.1), the south pointing arrow on the left and the east pointing arrow on the top are both epimorphisms. It follows that $\mathbf{im}\,(\mathscr{G} \to {}_V\mathscr{H}) = \mathbf{im}\,({}_V\mathscr{G} \to {}_V\mathscr{H}) = \mathbf{im}\,(\mathscr{H} \to {}_V\mathscr{H})$. Let $\mathscr{K} \subset {}_V\mathscr{H}$ be this common image.

Consider the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{F} & \xrightarrow{\;i\;} & \mathscr{G} & \xrightarrow{\;\pi\;} & \mathscr{H} & \longrightarrow & 0 \\ & & {\scriptstyle (5.1.3)}\big\downarrow & & \big\downarrow{\scriptstyle (5.1.3)} & & \big\downarrow & & \\ 0 & \longrightarrow & {}_V\mathscr{F} & \xrightarrow[\text{via } i]{} & {}_V\mathscr{G} & \longrightarrow & \mathscr{K} & \longrightarrow & 0 \\ & & & & & \searrow^{\text{via } \pi} & \big\downarrow\hookrightarrow & & \\ & & & & & & {}_V\mathscr{H} & & \end{array} \tag{5.1.5.2}$$

The composite of the two downward arrows in the third column is (5.1.3). From (5.1.5.2) and the fact that $\Gamma(X, {}_V\mathscr{G}) = \Gamma(V, \mathscr{G})$ and $\Gamma(X, {}_V\mathscr{H}) = \Gamma(V, \mathscr{H})$, we get a commutative diagram with exact rows:

$$\begin{array}{ccccccc} \Gamma(X, \mathscr{G}) & \xrightarrow{\pi(X)} & \Gamma(X, \mathscr{H}) & \xrightarrow{\;\delta\;} & \mathrm{H}^1(X, \mathscr{F}) & \longrightarrow & 0 \\ {\scriptstyle (5.1.3)}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle (5.1.3)} & & \\ \Gamma(V, \mathscr{G}) & \longrightarrow & \Gamma(X, \mathscr{K}) & \xrightarrow{\;\delta\;} & \mathrm{H}^1(X, {}_V\mathscr{F}) & \longrightarrow & 0 \\ & \searrow^{\text{via } \pi} & \big\downarrow\hookrightarrow & & & & \\ & & \Gamma(V, \mathscr{H}) & & & & \end{array} \tag{5.1.5.3}$$

We have used the fact that ${}_V\mathscr{G}$ is flasque to get the exactness in the lower horizontal row (see Proposition 5.1.4). Since the restriction $\Gamma(X, \mathscr{H}) \to \Gamma(V, \mathscr{H})$ factors through $\Gamma(X, \mathscr{K}) \subset \Gamma(V, \mathscr{H})$, $\beta|_V$ is an element of $\Gamma(X, \mathscr{K})$. Now $\beta|_V$ maps to the image of α in $\mathrm{H}^1(X, {}_V\mathscr{F})$ by the naturality of connecting homomorphisms (and since β maps to α). On the other hand $\beta|_V = \pi(V)(\gamma_V)$ and hence is in the image of the first arrow in the second row. From the exact sequence on the second row, it follows that the image of α in $\mathrm{H}^1(X, {}_V\mathscr{F})$ is zero. □

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