

LECTURE 1

Date of Lecture: September 21, 2021

The symbol $\mathcal{A}b$ will denote the category of abelian groups. The symbol $\hat{\diamond}$ is for flagging a cautionary comment or a tricky argument. It occurs in the margins and is Knuth's version of Bourbaki's "dangerous bend symbol".

In this lecture will fix a topological space X , and τ will denote its collection of open sets.

1. Presheaves and sheaves

1.1. Presheaves. A *presheaf of abelian groups* F on X is a collection of abelian groups $\{F(U) \mid U \in \tau\}$ together with group homomorphisms $\rho_U^V: F(V) \rightarrow F(U)$, one for every inclusion of open sets $U \subset V$, such that for every pair of inclusions of open sets $U \subset V \subset W$ the following diagram commutes:

$$\begin{array}{ccc} F(W) & & \\ \rho_V^W \downarrow & \searrow \rho_U^W & \\ F(V) & \xrightarrow{\rho_U^V} & F(U) \end{array}$$

An alternate definition is the following. Let \mathcal{C}_τ be the category whose objects are members of τ and

$$\text{Hom}_{\mathcal{C}_\tau}(U, V) = \begin{cases} \emptyset & \text{if } U \text{ is not a subset of } V \\ U \subset V & \text{otherwise.} \end{cases}$$

A presheaf F of abelian groups on X is a functor¹

$$F: \mathcal{C}_\tau^\circ \longrightarrow \mathcal{A}b.$$

Here \mathcal{C}_τ° denotes the "opposite category" of \mathcal{C}_τ . In other words F is an $\mathcal{A}b$ -valued *contravariant* functor on \mathcal{C}_τ .

Let V be an open set. An element s of $F(V)$ is called a *section of F over V* . A section of F over X is called a *global section* of F . If U is an open subset of V and s a section of F over V then we often write $s|_U$ for $\rho_U^V(s)$. Thus

$$s|_U := \rho_U^V(s).$$

The element $s|_U$ is called *the restriction of s to U* .

A *sheaf of abelian groups* F on X is a presheaf satisfying the following additional requirements for every open set U and every open cover $(U_\alpha)_{\alpha \in \Lambda}$ of U :

- (a) If s is a section over U such that $s|_{U_\alpha} = 0$ for every $\alpha \in \Lambda$, then $s = 0$.
- (b) If we have a family of sections $(s_\alpha)_{\alpha \in \Lambda}$ with $s_\alpha \in F(U_\alpha)$ such that $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$, then there exists $s \in F(U)$, necessarily unique by (a), such that $s|_{U_\alpha} = s_\alpha$ for every $\alpha \in \Lambda$.

¹By a functor we mean a covariant functor.

1.2. The value of a sheaf on an empty set. One can show, using [Problem 5 of HW 1](#) that if \mathcal{F} is a sheaf on X , then $\mathcal{F}(\emptyset) = 0$. To see this, let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of \emptyset . Note that all U_α are empty sets, as are the double intersections $U_{\alpha\beta} := U_\alpha \cap U_\beta$, $\alpha, \beta \in \Lambda$. By *loc.cit.*, the sequence of abelian groups

$$0 \rightarrow \mathcal{F}(\emptyset) \xrightarrow{\epsilon} \prod_{\alpha \in \Lambda} \mathcal{F}(U_\alpha) \xrightarrow{d^0} \prod_{\alpha, \beta \in \Lambda} \mathcal{F}(U_{\alpha\beta})$$

is exact, where ϵ is the “diagonal” map $s \mapsto (s|_{U_\alpha})_\alpha$ and the map d^0 is defined by $d^0((s_\alpha)_\alpha) = (\sigma_{\alpha\beta})_{\alpha, \beta}$ where $\sigma_{\alpha\beta} = s_\beta|_{U_{\alpha\beta}} - s_\alpha|_{U_{\alpha\beta}}$. We may in fact take the indexing set Λ to be the empty set, since the union over an empty collection is empty. Now the product of abelian groups over an empty index set is zero. Hence we get the exact sequence

$$0 \longrightarrow \mathcal{F}(\emptyset) \longrightarrow 0 \longrightarrow 0$$

which forces $\mathcal{F}(\emptyset)$ to equal 0. More generally, if one has a sheaf with values in a category enriching $\mathcal{A}b$ (e.g. the category of rings) then $\mathcal{F}(\emptyset)$ will be the terminal object in that category. The last remark is not too important, and so if you do not understand what that means, you should not worry too much. However, you are invited to look at [this discussion](#).

1.2.1. Here are some examples of presheaves, some of which are sheaves.

1. For each non-empty open U in X set $\mathbf{Z}(U) = \mathbf{Z}$ and for an inclusion of non-empty open sets $U \subset V$, let $\rho_U^V: \mathbf{Z}(V) \rightarrow \mathbf{Z}(U)$ be the identity map $\mathbf{1}_{\mathbf{Z}}$ on \mathbf{Z} . Let $\mathbf{Z}(\emptyset) = 0$, and let the restriction maps to $\mathbf{Z}(\emptyset)$ be the only possible map (namely zero). \mathbf{Z} is clearly a presheaf. If X is Hausdorff and has more than one point then \mathbf{Z} is not a sheaf. Indeed, in this case one can find two non-empty open sets, say U_1 and U_2 , which are disjoint and so if we take $s_i \in \mathcal{F}(U_i)$ to be the elements $s_1 = 1$ and $s_2 = 0$, then $s_1|_{U_1 \cap U_2} = s_2|_{U_1 \cap U_2} = 0$. However there is no element s in $\mathbf{Z}(U_1 \cup U_2) = \mathbf{Z}$ such that $s|_{U_1} = 1$ and $s|_{U_2} = 0$.
2. Let X be an open subset of \mathbf{R}^n . For every open set U of X , let $\mathcal{C}_X^\infty(U)$ be the \mathbf{R} -algebra of smooth (i.e. infinitely differentiable) functions on U . Then \mathcal{C}_X^∞ is a sheaf of \mathbf{R} -algebras on X with the maps ρ_U^V being the usual restriction maps.
3. Similarly, for an open subset D of \mathbf{C} , one can define the sheaf $\mathcal{O}_D^{\text{hol}}$ of holomorphic functions on D . Here $\mathcal{O}_D^{\text{hol}}(U)$ is the \mathbf{C} -algebra of holomorphic functions on U , and restrictions are the usual restrictions of functions.
4. More generally, one can define such a sheaf of holomorphic functions on a domain D in \mathbf{C}^n since the notion of a holomorphic function in several variables on an open subset of \mathbf{C}^n makes sense. One amongst many equivalent definitions is that $f: D \rightarrow \mathbf{C}$ is holomorphic if it is so in each variable separately (keeping other variables constant), i.e. for each $i = 1, \dots, n$ the map

$$\zeta \mapsto f(z_1, \dots, z_{i-1}, \zeta, z_{i+1}, \dots, z_n)$$

is holomorphic as ζ varies in a manner such that $(z_1, \dots, z_{i-1}, \zeta, z_{i+1}, \dots, z_n) \in D$. Remarkably, one does not need to add the condition that f is continuous, since that comes for free via a theorem. Once again, the sheaf of holomorphic functions on D is denoted $\mathcal{O}_D^{\text{hol}}$. The superscript “hol” is usually dropped when one is only dealing with several complex variables or complex manifolds. It is added when comparisons are made with sheaves in algebraic geometry as in the famous [GAGA](#) theorem of Serre.

1.3. Morphisms of presheaves. Let F and G be presheaves of abelian groups on X . A morphism $\varphi: F \rightarrow G$ of presheaves is a family of group homomorphisms,

$$\varphi(U): F(U) \rightarrow G(U),$$

one for each $U \in \tau$, such that for every inclusion $U \subset V$ of open sets, the following diagram commutes

$$\begin{array}{ccc} F(V) & \xrightarrow{\theta(V)} & G(V) \\ \rho_U^V \downarrow & & \downarrow \rho_U^V \\ F(U) & \xrightarrow{\theta(U)} & G(U) \end{array}$$

In other words, θ is a *map of functors* between F and G . A map of functors is also known as a *natural transformation* or as a *functorial map*.

Thus presheaves on X form a category which we denote \mathcal{Psh}_X .

Sheaves on X also form a category by defining a morphism of sheaves to be a morphism of presheaves. We denote this category by \mathcal{Sh}_X . It is a fully faithful subcategory of \mathcal{Psh}_X , meaning that given two sheaves \mathcal{F} and \mathcal{G} , then

$$(1.3.1) \quad \text{Hom}_{\mathcal{Psh}_X}(\mathcal{F}, \mathcal{G}) = \text{Hom}_{\mathcal{Sh}_X}(\mathcal{F}, \mathcal{G}).$$

1.3.2. One can similarly talk about a presheaf/sheaf of sets, of rings, algebras, For example a presheaf F of rings would be a presheaf of abelian groups, with the added requirements that each $F(U)$ be a ring, and that the restriction maps ρ_U^V be ring homomorphisms. I will leave it to you to work out the definition of a presheaf or sheaf of algebras over a ring R .

1.4. An alternate definition of manifolds. Let \mathcal{A} be a sheaf of commutative rings on X . Then the pair (X, \mathcal{A}) is called a *ringed space*. We will be studying ringed spaces in greater detail later. An isomorphism of ringed spaces $(Y, \mathcal{B}) \xrightarrow{\sim} (X, \mathcal{A})$ is a homeomorphism $\varphi: Y \rightarrow X$ such that the sheaf $\varphi_*\mathcal{B}$ on X , given by $(\varphi_*\mathcal{B})(U) = \mathcal{B}(\varphi^{-1}(U))$, is isomorphic to \mathcal{A} as a sheaf of rings. For example $(U, \mathcal{C}_U^\infty)$ is a ringed space for an open set U of \mathbf{R}^n . The ringed spaces we are interested in in this subsection are $(B_n, \mathcal{C}_{B_n}^\infty)$, where B_n is the unit open ball centred at the origin in \mathbf{R}^n , $(\Delta, \mathcal{O}_\Delta^{\text{hol}})$ where Δ is the unit disc centred at 0 in \mathbf{C} .²

One can define various kinds of manifolds using the notion of ringed spaces, and these definitions are equivalent to the “classical” definitions. For example, a smooth differentiable manifold of dimension n is a ringed space (M, \mathcal{A}) such that M is Hausdorff and second countable (equivalently M is Hausdorff and paracompact) and M can be covered by open sets U_α , $\alpha \in \Lambda$ such that $(U_\alpha, \mathcal{A}|_{U_\alpha})$ is isomorphic to $(B_n, \mathcal{C}_{B_n}^\infty)$ for every $\alpha \in \Lambda$. Here, $\mathcal{A}|_{U_\alpha}$ has an obvious meaning as sheaf of rings on U_α .

Similarly a Riemann surface is a ringed space (S, \mathcal{A}) can be defined as a ringed space with S Hausdorff and second countable space which is locally isomorphic to $(\Delta, \mathcal{O}_\Delta^{\text{hol}})$. The second countability assumption is not necessary if S is connected by a theorem of Rado.

I leave it to you to define a complex manifold of dimension n as a ringed space.

²Be warned, the proof of the equivalence with the classical definition is subtle, though not difficult.

1.5. **Stalks.** Let F be a presheaf³ on X and $x \in X$ a point. The *stalk* F_x of F at x

$$(1.5.1) \quad F_x := \varinjlim_{U \ni x} F(U)$$

where the partial order \prec on the set of open sets is $U \prec V$ if $U \supset V$. Note that if V is an open set containing x , then the stalk of $F|_V$ at x is the same as the stalk of F at x , where $F|_V$ has obvious meaning.

If $s \in F(U)$ and $x \in U$, then the image of s in F_x (under the natural map $F \rightarrow F_x$) is called the *germ of s at x* . Elements of F_x are often called the germs of sections of F at x . Indeed, by standard properties of direct limits, every element of F_x is the germ of some section of F over a neighbourhood of x .

For a section s of F over an open set U and a point x in U , we write s_x for the germ of s at x .

1.5.2. Let $a \in \mathbf{C}$. Then the stalk $\mathcal{O}_{\mathbf{C},a}^{\text{hol}}$ of \mathcal{O}^{hol} is the \mathbf{C} -algebra of power series of the form $\sum_{\nu=0}^{\infty} c_{\nu}(Z - a)^{\nu}$ with positive radius of convergence.

2. Étale spaces associated with presheaves

Throughout this section F is a presheaf on X .

2.1. **The space $\mathcal{E}(F)$.** Define

$$(2.1.1) \quad \mathcal{E}(F) := \coprod_{x \in X} F_x.$$

For $U \in \tau$ and $s \in F(U)$, define a subset $W(s, U)$ of $\mathcal{E}(F)$ by the formula

$$(2.1.2) \quad W(s, U) = \{s_x \mid x \in U\}$$

where we remind the reader that s_x is the germ of s at x . It is easy to see that $\{W(s, U)\}$ is a basis for a topology on $\mathcal{E}(F)$. Endow $\mathcal{E}(F)$ with this topology. Note that we have a natural map

$$(2.1.3) \quad \pi: \mathcal{E}(F) \longrightarrow X$$

sending a member of F_x to x . It is easy to check that π is a continuous map.

2.2. **The sheafification of F .** Let U be open in X . Set

$$(2.2.1) \quad F^+(U) := \{\sigma: U \rightarrow \mathcal{E}(F) \mid \sigma \text{ is continuous and } \pi \circ \sigma = \mathbf{1}_U\}$$

Recall that maps σ such that $\sigma \circ \pi$ is the identity are called sections of π in set theory. So $F^+(U)$ can be regarded as the collection of continuous sections of π over U . Using pointwise addition (and if need be, other operations), we see that $F^+(U)$ is an abelian group (or, ring etc., as the case may be).

It is clear that F^+ is a sheaf with restrictions having the usual meaning. The sheaf F^+ is called the *sheafification of F* . In your homework you will see that if F is a sheaf then F can be identified with F^+ . In fact, more generally (even if F is not a sheaf), we have a map of presheaves

$$(2.2.2) \quad \theta = \theta_F: F \longrightarrow F^+$$

defined as follows: For an open set U the map $\theta(U)$ is the map which sends $s \in F(U)$ to the map $\sigma_s: U \rightarrow \mathcal{E}(F)$, where $\sigma_s(x) = s_x$. One has to check that σ_s is

³From now on, for the rest of the lecture, sheaves and presheaves will be sheaves and presheaves of abelian groups unless otherwise stated.

continuous, and this is part of your homework. Moreover, you have been asked to show that if F is a sheaf then θ is an isomorphism.

In fact the assignment $F \mapsto F^+$ is a functor from $\mathcal{Psh}_X \rightarrow \mathcal{Sh}_X$ which has the following universal property (we will see this later in the course): If \mathcal{G} is a sheaf and $\varphi: F \rightarrow \mathcal{G}$ is a map of presheaves, then there is a unique map of sheaves $\tilde{\varphi}: F^+ \rightarrow \mathcal{G}$ such that $\varphi = \tilde{\varphi} \circ \theta_F$.