

## HW 6

**Due date:** Nov 22, 2021

**The inverse image of a sheaf.** Let  $f: X \rightarrow Y$  be a continuous map of topological spaces, and  $\mathcal{G} \in \mathcal{S}h_Y$ . Then the inverse image  $f^{-1}\mathcal{G}$  of  $\mathcal{G}$  is the sheaf on  $X$  obtained from the fibre-product  $X \times_Y \mathcal{E}(\mathcal{G})$  consisting of points  $(x, e) \in X \times \mathcal{E}(\mathcal{G})$  such that  $f(x) = \pi_{\mathcal{G}}(e)$ , where  $\pi = \pi_{\mathcal{G}}: \mathcal{E}(\mathcal{G}) \rightarrow Y$  is the standard projection map. The space  $X \times \mathcal{E}(\mathcal{G})$  is given the product topology, and  $X \times_Y \mathcal{E}(\mathcal{G})$  the subspace topology. It is easy to see that the natural map  $X \times_Y \mathcal{E}(\mathcal{G}) \rightarrow X$  makes  $X \times_Y \mathcal{E}(\mathcal{G})$  into an étale space over  $X$  and hence gives a sheaf, which we denote  $f^{-1}\mathcal{G}$ . Equivalently, consider the presheaf  $f^*\mathcal{G}$  given by  $U \mapsto \varinjlim_V \mathcal{G}(V)$  where the direct limit is taken over open subsets  $V$  of  $Y$  containing  $f(U)$ . Then  $f^{-1}\mathcal{G}$  is the sheafification of  $f^*\mathcal{G}$ . It is not hard to show that  $f^{-1}$  is a left adjoint to  $f_*$ , i.e. one has a bifunctorial isomorphism  $\text{Hom}_{\mathcal{S}h_X}(f^{-1}\mathcal{G}, \mathcal{H}) \xrightarrow{\sim} \text{Hom}_{\mathcal{S}h_Y}(\mathcal{G}, f_*\mathcal{H})$ . The (easy) details may be found in [https://www.cmi.ac.in/~pramath/random\\_notes/upper-lower-star](https://www.cmi.ac.in/~pramath/random_notes/upper-lower-star).

1. Let  $f: X \rightarrow Y$  be a continuous map of topological spaces.
  - (a) Show that  $f^{-1}$  is exact. Using this, show that  $f_*\mathcal{E}$  is an injective sheaf on  $Y$  if  $\mathcal{E}$  is injective on  $X$ .
  - (b) Given  $x \in X$  and  $y = f(x)$ , show that for any sheaf  $\mathcal{G}$  on  $Y$ , there is a functorial map  $\mathcal{G}_y \rightarrow (f^{-1}\mathcal{G})_x$ .
  - (c) Show that the a map of ringed spaces  $(f, f^\#): (X, \mathcal{A}) \rightarrow (Y, \mathcal{B})$  is equivalent to the data  $(f, f^\flat)$  with  $f^\flat: f^{-1}\mathcal{B} \rightarrow \mathcal{A}$  a map of sheaves of rings. Show further that if  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  are locally ringed spaces then  $(f, f^\#)$  is a map of locally ringed spaces if and only if for every  $x \in X$ , with  $y = f(x)$ , the composite  $\mathcal{B}_y \rightarrow (f^{-1}\mathcal{B})_x \xrightarrow{f_x^\flat} \mathcal{A}_x$  is a local homomorphism.

**Tensor product of sheaves of modules and the upper-star functor.** Let  $(X, \mathcal{O}_X)$  be a ringed space, and let  $\mathcal{F}, \mathcal{G} \in \text{Mod}_{\mathcal{O}_X}$ . We have a presheaf  $\mathcal{F} \otimes_{\mathcal{O}_X}^P \mathcal{G}$  given by  $U \rightsquigarrow \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$ . We define the tensor product  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$  to be the sheafification of  $\mathcal{F} \otimes_{\mathcal{O}_X}^P \mathcal{G}$ . Note that  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \in \text{Mod}_{\mathcal{O}_X}$ . Combining the universal property of sheafifications and the universal property of tensor products we see that there is a universal  $\mathcal{O}_X$ -bilinear map (i.e. over each open  $U$  is bilinear as a map of  $\mathcal{O}_X(U)$ -modules)  $\mathcal{F} \times \mathcal{G} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$  such that if  $\mathcal{F} \times \mathcal{G} \rightarrow \mathcal{H}$  is a bilinear map of  $\mathcal{O}_X$ -modules, there is a unique map  $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \rightarrow \mathcal{H}$  in  $\text{Mod}_{\mathcal{O}_X}$  such that our bilinear map factors as  $\mathcal{F} \times \mathcal{G} \xrightarrow{\text{universal}} \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \rightarrow \mathcal{H}$ .

If  $f: X \rightarrow Y$  is a map of ringed spaces, and  $\mathcal{G} \in \text{Mod}_{\mathcal{O}_Y}$ , we define  $f^*\mathcal{G}$  as  $f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ , where  $\mathcal{O}_X$  is an  $f^{-1}\mathcal{O}_Y$  algebra via  $f^\flat$ . Note that  $f^*$  is right exact, but need not be exact. It is easy to see that  $\text{Hom}_X(f^*\mathcal{G}, \mathcal{H}) \xrightarrow{\sim} \text{Hom}_Y(\mathcal{G}, f_*\mathcal{H})$

where  $\text{Hom}_X(-, ?)$  is the Hom in  $\text{Mod}_{\mathcal{O}_X}$ . Once again see [https://www.cmi.ac.in/~pramath/random\\_notes/upper-lower-star](https://www.cmi.ac.in/~pramath/random_notes/upper-lower-star) (at least for the case of schemes and quasi-coherent sheaves, though the proof generalises).

2. Let  $X = \text{Spec } B$ ,  $Y = \text{Spec } A$ ,  $f: X \rightarrow Y$  a map of schemes, and  $M$  an  $A$ -module. Show that  $f^* \widetilde{M}$  is the sheafification of  $M \otimes_A B$ .
  
3. Suppose  $X$  is a scheme and  $\mathcal{A}$  a sheaf of  $\mathcal{O}_X$ -algebras on  $X$  such that, as an  $\mathcal{O}_X$ -module,  $\mathcal{A}$  is quasi-coherent. Show that there is an  $X$ -scheme  $\mathbf{Spec}(\mathcal{A})$  such that the structure morphism  $\Phi: \mathbf{Spec}(\mathcal{A}) \rightarrow X$  is an affine map, and such that  $\Phi_* \mathcal{O}_{\mathbf{Spec}(\mathcal{A})}$  is canonically isomorphic to  $\mathcal{A}$ . Show further that the canonical map  $\Phi^*: \mathcal{O}_X \rightarrow v_* \mathcal{O}_{\mathbf{Spec}(\mathcal{A})}$  is, under this identification, the algebra map  $\mathcal{O}_X \rightarrow \mathcal{A}$ . Conversely, if  $\Psi: S \rightarrow X$  is an affine map of schemes, show that  $\mathbf{Spec}(\Psi_* \mathcal{O}_S)$  is canonically isomorphic to  $S$  and that under this identification, the structure map  $\mathbf{Spec}(\Psi_* \mathcal{O}_S) \rightarrow X$  agrees with  $\Psi$ . I would like to remind you that according to item 4 of 3.1.1 of [Lecture 13](#),  $\Psi_* \mathcal{O}_S$  is quasi-coherent on  $X$  since  $\Psi$  is an affine morphism.