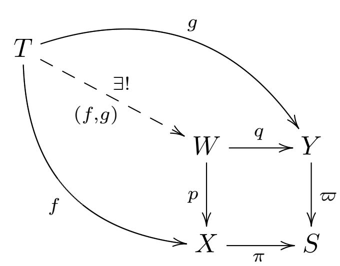
HW 5

Due date: Nov 10, 2021

1. Let $\mathscr{A}$ be an abelian category and $E^\bullet$ a bounded below injective complex of objects in $\mathscr{A}$ such that $\mathrm{H}^n(E^\bullet) = 0$ for $n \neq 0$ and $\mathrm{H}^0(E^\bullet) = A$. Show that there is a quasi-isomorphism $A \to E^\bullet$, where A is regarded as a complex in the usual way. [**Hint:** You may use the fact that if E is an injective object which is a subobject of an object X, then X must be of the form $X = E \oplus E'$. You do not have to prove this easy fact in the quiz. But see if you can prove it for yourself later.]

2. Let A_0 be a ring, and let A and B be A_0-algebras. Let $S = \operatorname{Spec} A_0$, $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$. Let $\pi\colon X \to S$ and $\varpi\colon Y \to S$ be the natural scheme maps. Show that there exists a scheme W together with scheme maps $p\colon W \to X$ and $q\colon W \to Y$ such that $\pi \circ p = \varphi \circ q$ and such that if we have a scheme T together with scheme maps $f\colon T \to X$, $g\colon T \to Y$ satisfying $\pi \circ f = \varpi \circ g$, then there exists a unique map $(f, g)\colon T \to W$ satisfying $p \circ (f, g) = f$ and $q \circ (f, g) = g$.

In other words, given a commutative diagram with solid arrows as below, the broken arrow can be filled in exactly one way to make the resulting diagram commute:



Comment: The datum (W, p, q) is clearly unique up to unique isomorphism because of the universal property the datum has. It is usually denoted $X \times_S Y$. As is common in mathematics, the symbol is used for the datun (W, p, q) as well as for the scheme W. The scheme or the datum $X \times_S Y$ is called the *fibre product* of X and Y over S.