HW 4

Due date: Nov 3, 2021

As always, "map" is used for "morphism". In particular a "map of complexes" is either chain map pr a co-chain map, depending on whether the complexes in question are chain complexes or co-chain complexes. For problems involving an abelian category $\mathscr{A}$, you may, if you feel like, assume $\mathscr{A} = \mathrm{Mod}_A$, the category of modules of a ring A.

The symbols $\mathscr{Psh}_X$ and $\mathscr{Sh}_X$ are as in the lectures.

For a ringed space $(X, \mathscr{O}_X)$, the symbol $\mathrm{Mod}_{\mathscr{O}_X}$ will denote the category of $\mathscr{O}_X$-modules.

If $\mathfrak{U} = (U_\alpha)$ is an open cover of a topological space X, and V is an open subset of X, then $\mathfrak{U} \cap V$ denotes the open cover $(U_\alpha \cap V)$ of V.

Quasi-coherent sheaves on affine schemes. Recall that if A is a ring and $M \in \mathrm{Mod}_A$, then $\widetilde{M}$ is the sheaf of $\mathscr{O}_{\mathrm{Spec}\,A}$-modules defined by $D(f) \mapsto M_f$, $f \in A$, with restrictions given by further localisation. Sometimes it is useful to specify the ring A (e.g. if $A' \to A$ is a ring homomorphism, so that M is an A'-module and an A-module). In that case we use the symbol $\widetilde{M}_A$. Recall that an $\mathscr{O}_X$-module $\mathscr{F}$ on $X = \mathrm{Spec}\,A$ is said to be *quasi-coherent* if $\mathscr{F}$ is isomorphic to $\widetilde{M}$ as an $\mathscr{O}_X$-module for some $M \in \mathrm{Mod}_A$.

For problems in this section, fix a ring A and let $X = \mathrm{Spec}\,A$. For $f \in A$, $D(f)$ is identified with the scheme $\mathrm{Spec}\,A_f$. Note that $\mathscr{O}_{D(f)} = \mathscr{O}_X|_{D(f)}$

1. (a) Let $f \in A$ and $M \in \mathrm{Mod}_A$. Show that $(\widetilde{M}_A)|_{D(f)} = (\widetilde{M_f})_{A_f}$. Conclude that if $\mathscr{F}$ is quasi-coherent on X, then $\mathscr{F}|_{D(f)}$ is quasi-coherent for each $f \in A$.

(b) For a map of rings $A \to B$, with $Y = \mathrm{Spec}\,B$ and $\alpha\colon Y \to X$ the map of schemes induced by $A \to B$, show that $\alpha_*(\widetilde{M}_B) = \widetilde{M}_A$, for $M \in \mathrm{Mod}_B$. Here α_* is the direct image functor defined in (1.2.1) of Lecture 3. In particular, deduce that if $f \in A$ and $i\colon D(f) \to X$ is the natural open inclusion, then $i_*((\widetilde{M_f})_{A_f}) = (\widetilde{M_f})_A$.

2. Suppose we have elements $f_0, \ldots, f_d \in A$ such that $X = \cup_{i=0}^d D(f_i)$ (equivalently $\langle f_0, \ldots, f_d \rangle = A$). Let $\mathfrak{U}$ be the ordered open cover $(D(f_i))_{i=0}^d$ of X. Let $\mathscr{F}$ be an $\mathscr{O}_X$-module such that $\mathscr{F}|_{D(f_i)}$ is quasi-coherent on the affine scheme $D(f_i)$ for $i = 0, \ldots, d$.

(a) Show that the Čech complex $C^\bullet(\mathfrak{U}, \mathscr{F})$ is a complex of A-modules.

(b) Let $g \in A$. Show that the localisation $C^\bullet(\mathfrak{U}, \mathscr{F})_g$ of the Čech complex of A-modules $C^\bullet(\mathfrak{U}, \mathscr{F})$ at g is the Čech complex of $\mathscr{F}|_{D(g)}$ with respect to the cover $\mathfrak{U} \cap D(g)$.

(c) Show that $\mathscr{F}$ is quasi-coherent. [**Hint:** Let $M = \Gamma(X\,\mathscr{F})$. Localise the exact sequence $0 \to M \to C^0(\mathfrak{U}, \mathscr{F}) \to C^1(\mathfrak{U}, \mathscr{F})$ at various $g \in A$ and compute $\mathscr{F}(D(g))$. Use the fact that A_g is a flat A-algebra, whence localisation at g is an exact functor.]

3. Let $\mathfrak{U} = (U_\alpha)$ be an affine open cover[1] of X such that $\mathscr{F}|_{U_\alpha}$ is quasi-coherent for every α. Show that $\mathscr{F}$ is quasi-coherent. [**Hint:** You might need to use the fact that X is quasi-compact since it is an affine scheme.]

4. Suppose Z is a scheme, $\mathscr{F}$ an $\mathscr{O}_Z$-module, and $\mathfrak{U} = (U_\alpha)$ an affine open cover of Z such that $\mathscr{F}|_{U_\alpha}$ is quasi-coherent for every α. Let $X = \operatorname{Spec} A$ be an affine open subscheme of Z.[2] Show that $\mathscr{F}|_X$ is quasi-coherent. [**Hint:** Cover X by affine open subschemes on which $\mathscr{F}$ is quasi-coherent and use the previous problem.]

$\mathscr{O}_X$-modules on a scheme. In this section we fix a scheme X, not necessarily affine. Let $\mathscr{F} \in \mathrm{Mod}_{\mathscr{O}_X}$. We say $\mathscr{F}$ is a *quasi-coherent $\mathscr{O}_X$-module* if there exists an affine open cover $\mathfrak{U}$ of X such that $\mathscr{F}|_U$ is quasi-coherent for each $U \in \mathfrak{U}$. Equivalently (by Problem **4**.) $\mathscr{F}$ is quasi-coherent if $\mathscr{F}|_U$ is quasi-coherent for every affine open subscheme of X.

Fix a scheme X.

5. Let X be affine, say $X = \operatorname{Spec} A$. Let $\mathscr{B} = \{D(f)\}$ be the standard base for the topology on X. Let $\mathscr{F}$ be a $\mathrm{Mod}_{\mathscr{O}_X}$-module and $M = \Gamma(X, \mathscr{F})$.

(a) Show that the natural map $M_f \to \mathscr{F}(D(f))$, for $f \in A$, arising from the universal property of localisation, gives a map of $\mathscr{B}$-sheaves $\widetilde{M}|_{\mathscr{B}} \to \mathscr{F}|_{\mathscr{B}}$.

(b) Let $\varphi_{\mathscr{F}} \colon \widetilde{M} \to \mathscr{F}$ be the resulting map of sheaves. It is clearly a map of $\mathscr{O}_X$-modules (you don't have to prove this). Show that $\varphi_{\mathscr{F}}$ is functorial in $\mathscr{F}$. In greater detail, writing $M_{\mathscr{F}}$ for $\Gamma(X, \mathscr{F})$, show that given a map $\mathscr{F} \to \mathscr{G}$ in $\mathrm{Mod}_{\mathscr{O}_X}$, the following diagram commutes

$$\begin{array}{ccc} \widetilde{M_{\mathscr{F}}} & \xrightarrow{\text{via } \varphi} & \widetilde{M_{\mathscr{G}}} \\ \downarrow{\scriptstyle \varphi_{\mathscr{F}}} & & \downarrow{\scriptstyle \varphi_{\mathscr{G}}} \\ \mathscr{F} & \xrightarrow[\varphi]{} & \mathscr{G} \end{array}$$

6. Let

$$0 \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_2 \longrightarrow \mathscr{F}_3 \longrightarrow 0$$

be a short exact sequence of $\mathscr{O}_X$-modules.

(a) Show that if $\mathscr{F}_2$ and $\mathscr{F}_3$ are quasi-coherent, then so is $\mathscr{F}_1$.

(b) Show that if $\mathscr{F}_1$ and $\mathscr{F}_2$ are quasi-coherent, then so is $\mathscr{F}_3$.

(c) Show that if $\mathscr{F}_1$ and $\mathscr{F}_3$ are quasi-coherent, then so is $\mathscr{F}_2$.

Hint: Since quasi-coherence is a local property, without loss of generality assume thar X is affine. Let $M_i = \Gamma(X\,\mathscr{F}_i)$. Apply Problem **5**. For part (a) use the fact that $\Gamma(X, -)$ is left exact and that $\widetilde{(-)}$ is exact. For parts (b) and (c) use the fact that on an affine scheme Z any short exact sequence of sheaves

[1] i.e. an open cover such that every $(U_\alpha, \mathscr{O}_X|_{U_\alpha})$ is an affine scheme.

[2] This means X is open in Z and $\mathscr{O}_X = \mathscr{O}_Z|_X$.

$0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$, with $\mathscr{A}$ quasi-coherent, gives us a short exact sequence of abelian groups $0 \to \mathscr{A}(Z) \to \mathscr{B}(Z) \to \mathscr{C}(Z) \to 0$. We will prove this result later, and it rests on the fact that that the cohomology modules of quasi-coherent sheaves on affine schemes vanish. We don't need $\mathscr{B}$ or $\mathscr{C}$ to be in $\mathrm{Mod}_{\mathscr{O}_X}$ for the validity of this statement on exact sequences of global sections on affine schemes. The following commutative diagram with exact rows may be useful.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widetilde{M_1} & \longrightarrow & \widetilde{M_2} & \longrightarrow & \widetilde{M_3} & & \\ & & \downarrow{\scriptstyle \varphi_{\mathscr{F}_1}} & & \downarrow{\scriptstyle \varphi_{\mathscr{F}_2}} & & \downarrow{\scriptstyle \varphi_{\mathscr{F}_3}} & & \\ 0 & \longrightarrow & \mathscr{F}_1 & \longrightarrow & \mathscr{F}_2 & \longrightarrow & \mathscr{F}_3 & \longrightarrow & 0 \end{array}$$

You will have to argue that the top row is exact. And that sometimes one can put an arrow to 0 on the right of that row, and retain exactness.

The d-uple embedding. Please look at your tutorial notes for the intuitive definition of the d-uple embedding of a projective space into another projective space. This set of exercises is meant to make that rigorous. For a graded ring $S = \bigoplus_n S_n$, let S_+ be the ideal $S_+ = \bigoplus_{n>0} S_n$. For $d \geq 1$ we denote by $S^{(d)}$ be the graded ring given by $(S^{(d)})_n = S_{nd}$. It is well known that if $\phi\colon R \to S$ is a map of graded rings, $X = \mathrm{Proj}(S)$, $Y = \mathrm{Proj}(R)$, and $U(\phi) = \bigcup_{f \in R_+} D_+(\phi(f)) \subset X$, then we have a canonical map

$$r_\phi\colon U(\phi) \to Y.$$

In greater detail, the ring homomorphisms $R_{(f)} \to S_{(\phi(f))}$ give us maps $D_+(\phi(f)) \to D_+(f)$ which patch as f varies over R_+, to give r_ϕ. You may assume this easily provable fact. It is in this sense that we sometimes say that $\mathrm{Proj}(-)$ is functorial.

Note that the inclusion $S^{(d)} \subset S$ is *not* a graded ring map.

By a *closed embedding* of schemes we mean a map of schemes $j\colon Y \to Z$ such that the topological map $Y \to j(Y)$ induced by j is a homeomorphism, $j(Y)$ is a closed subset of Z, and the map $j^\#\colon \mathscr{O}_Y \to j_*\mathscr{O}_Z$ is a surjective map of sheaves (i.e. a map of sheaves whose cokernel sheaf is zero), and the kernel of $j^\#$ is quasi-coherent. For affine schemes this amounts to a surjective map of rings $A \to B$, with the scheme structure on $V(I)$ being provided by $\mathrm{Spec}\, B$, where $I = \ker(A \to B)$.

For us the projective space $\mathbb{P}^n_k$ over a field k is $\mathbb{P}^n_k = \mathrm{Proj}(k[T_0, \ldots, T_n])$.

7. Show that we have an isomorphism $\Psi\colon X \xrightarrow{\sim} Y$ where $X = \mathrm{Proj}(S)$ and $Y = \mathrm{Proj}(S^{(d)})$. Heed the warning given above. [**Hint:** Let $R = S^{(d)}$. The inclusion $R \subset S$ induces a scheme map $j\colon \mathrm{Spec}\, S \to \mathrm{Spec}\, R$. Show that if $\mathfrak{p}$ is a graded prime ideal of S then $j(\mathfrak{p}) = \mathfrak{p} \cap R$ is a graded prime ideal of R. Show that if $f \in S_+$ is homogeneous and $f \notin \mathfrak{p}$, then $f^d \in R_+$ and $f^d \notin j(\mathfrak{p})$. Show further that if $\mathfrak{q} \subset R$ is a graded prime ideal and $I := \mathfrak{q}S$ the ideal of S generated by elements of $\mathfrak{q}$, then I is homogeneous, $\mathfrak{q} = I \cap R$, $\mathfrak{p} := \sqrt{I}$ is a graded prime ideal of S, $j(\mathfrak{p}) = \mathfrak{q}$, and if $\mathfrak{q}$ does not contain R_+ then $\mathfrak{p}$ does not contain S_+. Conclude that j induces a homeomorphism $i\colon X \to Y$. Next show that $S_{(f)} \cong R_{(f^d)}$ for f homogeneous. Conclude that one has an isomorphism $\mathscr{O}_Y \to i_*\mathscr{O}_X$.]

8. As above, let d be a positive integer. Let k be a field. Consider the subring $S = k[T_0^{\mu_0} \dots T_n^{\mu_d} \mid \sum_{i=0}^{n} \mu_i = d]$ of the ring $k[T_0, \dots, T_n]$ and give R the grading which gives each monomial $\mathbf{T}^{\boldsymbol{\mu}}$ degree 1 (in other words, $S = k[\mathbf{T}]^{(d)}$, with $k[\mathbf{T}]$ being the the standard graded ring of polynomials in $(n+1)$-variables). Let R be the polynomial ring in $\binom{n+d}{n}$ variables and write R as $R = k[Z_{\boldsymbol{\mu}} \mid \sum_{i=0}^{n} \mu_i = d]$, where the μ_i are non-negative integers and the $Z_{\boldsymbol{\mu}}$ are indeterminates (we are using the well known combinatorial fact that the number of $\boldsymbol{\mu}$ of the kind we specified is $\binom{n+d}{d}$). Show that the graded ring map $\phi\colon R \to S$ given by $Z_{\boldsymbol{\mu}} \to \mathbf{T}^{\boldsymbol{\mu}}$ gives a closed embedding of $\mathbb{P}_k^n \hookrightarrow \mathbb{P}_k^N$ where $N = \binom{n+d}{d} - 1$. [**Hint:** First show that $U(\phi)$ defined in the beginning of this section is $\mathrm{Proj}(S)$. Then show that $R_{(Z_{\boldsymbol{\mu}})} \to S_{(\mathbf{T}^{\boldsymbol{\mu}})}$ is surjective.]

It is worth pointing out that the field k above is not algebraically closed. In our course we will de-emphasise algebraic closure (we will sometimes use it, but generally it won't be necessary). In fact note that the problem really did not require k to be a field. It could have been a ring, and one would then be getting results for W-schemes, where $W = \operatorname{Spec} k$. One can go one step further and actually see that everything goes through for arbitrary W, not necessarily affine W. All this is food for thought, not to be submitted.