HW 3

Due date: Oct 25, 2021

As always, "map" is used for "morphism". In particular a "map of complexes" is either chain map pr a co-chain map, depending on whether the complexes in question are chain complexes or co-chain complexes. For problems involving an abelian category $\mathscr{A}$, you may, if you feel like, assume $\mathscr{A} = \mathrm{Mod}_A$, the category of modules of a ring A.

The symbols $\mathscr{Psh}_X$ and $\mathscr{Sh}_X$ are as in the lectures.

The functor Γ. Let X be a topological space. The association $F \mapsto F(X)$, as F varies in $\mathscr{Psh}_X$, gives us a functor, the so called *global sections functor*, from $\mathscr{Psh}_X$ to $\mathscr{Ab}$. This functor is denoted $\Gamma(X, -)$. Thus $\Gamma(X, F) = F(X)$ for $F \in \mathscr{Psh}_X$. For an open subset U of X, the convention is to use the short hand $\Gamma(U, F)$ instead of $\Gamma(U, F|_U)$ for a presheaf F on X. We will follow that convention.

1. Let $(f, f^\#)\colon (X, \mathscr{O}_X) \to (Y, \mathscr{O}_Y))$ be a morphism in the category of *locally ringed spaces*, and suppose Y is an affine scheme,[1] say $Y = \operatorname{Spec} A$. Let $x \in X$, and consider the map of rings $A \to \mathscr{O}_{X,x}$ given by the composite

$$A = \Gamma(Y, \mathscr{O}_Y) \xrightarrow{\Gamma(Y, f^\#)} \Gamma(X, \mathscr{O}_X) \longrightarrow \mathscr{O}_{X,x}$$

Let $\mathfrak{m}_x$ be the maximal ideal of $\mathscr{O}_{X,x}$ and $\mathfrak{p}$ the prime ideal in A which is the inverse image of $\mathfrak{m}_x$. Let $y \in Y$ be the point corresponding to $\mathfrak{p}$. Show that $f(x) = y$. [**Note:** We are not assuming X is a scheme. It is merely a locally ringed space.]

2. Let X be a locally ringed space and let $f \in \Gamma(X, \mathscr{O}_X)$. For $x \in X$, let $\mathfrak{m}_x$ be the maximal ideal of $\mathscr{O}_{X,x}$. Fix $x \in X$. Show that if the image of f in $\mathscr{O}_{X,x}$ does not lie in $\mathfrak{m}_x$ then there is an open neighbourhood U of x such that $f|_U$ is invertible in $\Gamma(U, \mathscr{O}_X)$.

Direct Limits. For a quick recall of the definitions and the existence of direct limits, look up these notes.

3. Let A be a ring, $t \in A$ an element, and Q be an A-module. Let $(Q_n)_{n\geq 0}$ be the directed system with $Q_n = Q$ for all n and with transition maps $\mu_{m,n}$ given by $x \mapsto t^{n-m}x$ for $m \leq n$. Show that

$$\varinjlim_n Q_n = Q_t$$

where Q_t is the localisation of Q at the multiplicative system $\{1, t, t^2, \dots, t^n, \dots\}$. [**Hint:** Let $\nu_n\colon Q_n \to Q_t$ be the map $x \mapsto x/t^n$. Check that $\nu_n \circ \mu_{m,n} = \nu_m$ for $m \leq n$. Show that the resulting map $\nu\colon \varinjlim_n Q_n \to Q_t$ is an isomorphism.]

[1] i.e. $(Y, \mathscr{O}_Y)$ is an affine scheme.

Koszul and Čech. In this subsection A is a ring, $\boldsymbol{t} = (t_1, \ldots, t_d)$ a d-tuple of elements in A, I the ideal $\langle t_1, \ldots, t_d\rangle$ generated by the t_i, X the affine scheme $X = \operatorname{Spec} A$, Z the closed subset $V(I)$ of X, $U = X \smallsetminus Z$.

For $f \in A$, we use the more standard notation $D(f)$ for the open set X_f we defined in §2.2 of Lecture 2.

Let U_i be the open subscheme $U_i = \operatorname{Spec} A_{t_i} = D(t_i)$ of X, $i = 1, \ldots, d$ and $\mathfrak{U}$ the family of open sets $\{U_i\}_{i=1,\ldots,d}$. Note that $\mathfrak{U}$ is an open cover of U.

Fix an A-module M. Set

$$K^\bullet_\infty = K^\bullet_\infty(\boldsymbol{t},\, M) := \varinjlim_{\boldsymbol{\nu}\in\mathbf{N}^d} K^\bullet(\boldsymbol{t}^{\boldsymbol{\nu}},\, M),$$

where $(K^\bullet(\boldsymbol{t}^{\boldsymbol{\nu}},\, M))_{\boldsymbol{\nu}}$ is the direct system defined in (3.1.2) of these supplementary notes on complexes. The p^{th} differential of this complex will be denoted d^p_∞.

4. Let $C^\bullet = C^\bullet(\mathfrak{U}, \widetilde{M}|_U)$ where $\widetilde{M}$ is the sheaf on X defined by the $\mathscr{B}$-sheaf $\Gamma(D(f), \widetilde{M}) = M_f$ for $f \in A$, and $\mathscr{B} = \{D(g)\}_{g\in A}$ the standard base for the topology on X. Show that
   (a) Show that $C^p = K^{p+1}_\infty$, $p \geq 0$.
   (b) Show that $d^p_C = d^{p+1}_\infty$, $p \geq 0$.
   (c) Show that if $\boldsymbol{t}$ is an M-sequence then $\mathrm{H}^p(K^\bullet_\infty) = 0$ for $p \neq d$. [**Hint:** Use the fact that direct limits preserve exactness and hence commute with cohomology.]

   Hint: Use Problem 3.

5. Let $R = A[T_0, \ldots, T_n]$ be the polynomial ring in $n+1$ variables over A. Let $\mathbb{A}^{n+1}_A = \operatorname{Spec} R$, J the ideal generated by $T_0, \ldots, T_n$, V the complement of $V(J)$ in $\mathbb{A}^{n+1}_A$, and $V_i = D(T_i)$, $i = 0, \ldots, n$. Let $\mathfrak{V} = \{V_i\}_{i=0}^n$ be the cover of V given by the V_i. Show that
   (a) $\check{\mathrm{H}}^i(\mathfrak{V},\, \mathscr{O}_V) = 0$ for $i \neq 0, n$.
   (b) $\check{\mathrm{H}}^0(\mathfrak{V},\, \mathscr{O}_V) = R$.
   (c) $\check{\mathrm{H}}^n(\mathfrak{V},\, \mathscr{O}_V)$ is the R-module P of *inverse polynomials* in $T_0, \ldots, T_n$, i.e. as an A-module P is the direct sum of the free rank one A-modules $A_{\boldsymbol{\nu}} = A \cdot T_0^{\nu_0} \ldots T_n^{\nu_n}$ with each $\nu_i < 0$ and the R-module structure is given as follows: For a monomial $\mathbf{T}^{\boldsymbol{\mu}} := T_0^{\mu_0} \ldots T_n^{\mu_n}$ in $A[T_0, \ldots, T_n]$ and an A-basis element $\mathbf{T}^{\boldsymbol{\nu}} := T_0^{\nu_0} \ldots T_n^{\nu_n}$ of P (in the just mentioned direct sum decomposition of P) we have

$$\mathbf{T}^{\boldsymbol{\mu}} \cdot \mathbf{T}^{\boldsymbol{\nu}} = \begin{cases} T_0^{\mu_0+\nu_0} \ldots T_n^{\mu_n+\nu_n} & \text{if } \mu_i + \nu_i < 0 \text{ for } i = 0, \ldots, n \\ 0 & \text{otherwise} \end{cases}$$

When you solve Problem 5 (and it is not difficult), you would have essentially proved the main result of this course, namely the so-called *Cohomology of Projective Space* which is in Chapter III, p. 225 of Hartshorne. This is a short cut to that result.