**HW 2**

**Due date:** Oct 11, 2021

As always, "map" is used for "morphism". In particular a "map of complexes" is either chain map pr a co-chain map, depending on whether the complexes in question are chain complexes or co-chain complexes. For problems involving an abelian category $\mathscr{A}$, you may, if you feel like, assume $\mathscr{A} = \mathrm{Mod}_A$, the category of modules of a ring $A$. For a complex $C^\bullet$, $Z^p(C^\bullet)$ is the kernel of $d_C^p$ and $B^p(C^\bullet)$ is the image of $d_C^{p-1}$, i.e. if we are dealing with $\mathscr{A} = \mathrm{Mod}_A$, $Z^p(C^\bullet)$ is the module of $p$-cocycles of $C^\bullet$ and $B^p(C^\bullet)$ is the module of $p$-coboundaries of $C^\bullet$. As always, the $p^{\text{th}}$ cohomology $\mathrm{H}^p(C^\bullet)$ of $C^\bullet$ is the "quotient":

$$\mathrm{H}^p(C^\bullet) := Z^p(C^\bullet)/B^p(C^\bullet).$$

Please look at https://www.cmi.ac.in/~pramath/AGI/notes/CechNotes.pdf for various definitions involving Čech complexes and the $\mathrm{Hom}^\bullet$ complexes.

**Homotopies.** Let $\alpha\colon C^\bullet$ and $D^\bullet$ be two complexes in an abelian category $\mathscr{A}$. A map of complexes $\alpha\colon C^\bullet \to D^\bullet$ is said to be *homotopic to* 0 if there exist maps $k^p\colon C^p \to D^{p-1}$, $p \in \mathbf{Z}$, such that $\alpha^p = d_D^{p-1} \circ k^p + k^{p+1} \circ d_C^p$ for every $p \in \mathbf{Z}$. In this case we write $\alpha \sim 0$. Note that if $\alpha \sim 0$ then $-\alpha \sim 0$. Two maps $\alpha, \beta\colon C^\bullet \to D^\bullet$, are said to be homotopic to each other if $\alpha - \beta \sim 0$. Homotopy is clearly an equivalence relation between maps of complexes.

**1**. Show that if $\alpha \sim \beta$ then $\mathrm{H}^p(\alpha) = \mathrm{H}^p(\beta)$ for all $p \in \mathbf{Z}$.

**2**. Let $T^\bullet = \mathrm{Hom}^\bullet_{\mathscr{A}}(C^\bullet, D^\bullet)$.
  **(a)** Show that $Z^0(T^\bullet)$ is the group of maps of complexes from $C^\bullet$ to $D^\bullet$.
  **(b)** Show that $B^0(T^\bullet)$ is the group of maps of complexes from $C^\bullet$ to $D^\bullet$ which are homotopic to zero.

**The sheaf Čech complex.** Let $X$ be a topological space, and $\mathfrak{U} = \{U_\alpha\}_{\alpha\in\Lambda}$ an open cover of $X$ with $\Lambda$ totally ordered. For any open set $V$ of $X$, set $\mathfrak{U} \cap V := \{U_\alpha \cap V\}$. Fix $p \in \{0, 1, 2, \ldots, n, \ldots\}$. If $C^\bullet(\mathfrak{U}, \mathscr{F})$ denotes the Cech complex of a sheaf of $\mathscr{F}$, let $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ be the presheaf given by $V \mapsto C^p(\mathfrak{U} \cap V, \mathscr{F}|_V)$, $V$ open in $X$. It is easy to check that $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is a sheaf and that the coboundaries in the Cech complex restrict well to open subsets, and hence we have a complex, the so called *sheaf Cech complex*, $\mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ as well as a map $\mathscr{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$.

**3**. Show that the natural map $\mathscr{F} \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})$ is such that the induced map $\mathscr{F}_x \to \mathscr{C}^\bullet(\mathfrak{U}, \mathscr{F})_x$ on stalks is a quasi-isomorphism for every $x \in X$. [**Hint:** Find a homotopy between the zero map and the identity map on the augumented complex

$$0 \to \mathscr{F}_x \to \mathscr{C}^0(\mathfrak{U}, \mathscr{F})_x \to \mathscr{C}^1(\mathfrak{U}, \mathscr{F})_x \to \cdots \to \mathscr{C}^n(\mathfrak{U}, \mathscr{F})_x \to \ldots.$$

In greater detail, for simplicity assume that $x \in U_{\alpha^*}$, where $\alpha^*$ is the smallest element in the well-ordered set $\Lambda$. Let $\boldsymbol{\xi} \in \mathscr{C}^p(\mathfrak{U}, \mathscr{F})_x$. Then $\boldsymbol{\xi}$ is represented by a section $\sigma \in C^p(\mathfrak{U} \cap V, \mathscr{F})$ for some neighbourhood $V$ of $x$. Without loss of generality, we may assume $V \subset U_{\alpha^*}$. Let $\kappa^p(\sigma) \in C^{p-1}(\mathfrak{U} \cap V, \mathscr{F})$ be the element whose $(\alpha_0, \ldots, \alpha_{p-1})^{\text{th}}$ component, with $\alpha_0 < \cdots < \alpha_{p-1}$, is $\sigma_{\alpha^*\alpha_0\ldots\alpha_{p-1}} \in \mathscr{F}(U_{\alpha^*\alpha_0\ldots\alpha_{p-1}} \cap V) = \mathscr{F}(U_{\alpha_0\ldots\alpha_{p-1}} \cap V)$. Let $k^p(\boldsymbol{\xi})$ be the germ of $\kappa^p(\sigma)$ at $x$. Check everything is well-defined and $\{k^p\}$ is the required homotopy for such an $x$. What would you do if $x \notin U_{\alpha^*}$?]

**The punctured affine plane.** Let $k$ be a field, $A = k[S, T]$, the polynomial ring over $k$ in two variables, $\mathfrak{m}_\circ$ the maximal ideal $\langle S, T\rangle$ of $A$, $\mathbb{A}^2_k = \operatorname{Spec} A$, $X = \mathbb{A}^2_k \smallsetminus \{\mathfrak{m}_\circ\}$, $U_0 = \operatorname{Spec} A_S$, $U_1 = \operatorname{Spec} A_T$. In geometric terms, $\mathbb{A}^2_k$ is the affine plane over $k$, $\mathfrak{m}_\circ$ the origin of this affine plane, $X$ the plane punctured at the origin, $U_0$ the affine plane minus the $T$-axis, $U_1$ the affine plane minus the $S$-axis. Note that $\mathfrak{U} = \{U_0, U_1\}$ is an open cover of the punctured plane $X$.The Čech complex $C^\bullet(\mathfrak{U}, \mathscr{O}_X)$ is clearly the complex

$$0 \longrightarrow A_S \oplus A_T \xrightarrow{\ d\ } A_{ST} \longrightarrow 0$$

where $d(a, b) = \frac{b}{1} - \frac{a}{1}$. The grading is such that $A_S \oplus A_T$ is in the $0^{\text{th}}$-place.

**4**. Show that $\check{\mathrm{H}}^1(\mathfrak{U}, \mathscr{O}_X)$ can be identified with the module of *inverse polynomials in $S$ and $T$*, i.e., the $A$-module which as a $k$-vector space is generated by the linearly independent elements $S^\mu T^\nu$ where $\mu, \nu < 0$, and whose $A$-module structure is given by $S^m T^n (S^\mu T^\nu) = S^{m+\mu} T^{n+\nu}$ if $m + \mu$ and $n + \nu$ are both negative, and is zero otherwise.

**Varieties and Schemes.** The following problems deal with some material which will be covered soon.

**5**. Let $t$ be the functor $t : \mathbb{V}\mathrm{ar}_{/k} \to \mathbb{S}\mathrm{ch}_{/k}$ from the category of varieties to the category of schemes over an algebraically closed field $k$. Show that for any two varieties $V, W$ over $k$, the natural map

$$\operatorname{Hom}_{\mathbb{V}\mathrm{ar}_{/k}}(V, W) \longrightarrow \operatorname{Hom}_{\mathbb{S}\mathrm{ch}_{/k}}(t(V), t(W))$$

is bijective.

**6**. Let $S$ be a graded ring and $f$ be a homogenous element of $S_+$. For any homogenous ideal $\mathfrak{a} \subseteq S$, let $\phi(\mathfrak{a}) = \mathfrak{a}S_f \cap S_{(f)}$. Show that $\phi$ gives a bijective map from $D_+(f)$ to $\operatorname{Spec} S_{(f)}$.