HW 1

Due date: Sep 29, 2021

As always, "map" is used for "morphism".

Étale spaces. For a presheaf F on a topological space, we will use the notations we used in class. Thus $\mathscr{E}(F)$ is the topological space associated with F, $\pi\colon \mathscr{E}(F) \to X$ the natural map, F^+ the sheafification of F etc.

In what follows, X is a topological space, and F a presheaf on X.

1) Show that $\pi\colon \mathscr{E}(F) \to X$ is a local homeomorphism.

2) Show that if U is open in X and $s \in F(U)$, and for $x \in U$, s_x the germ of s at x,[1] then the map

$$\sigma_s\colon U \to \mathscr{E}(F) = \coprod_{x\in X} F_x$$

given by $x \mapsto s_x$, $x \in U$, is a continuous map.

Recall that the natural map $\theta(=\theta_F)\colon F \to F^+$ is the map defined on every open set U of X by $s \mapsto \sigma_s$ with the notation as above.

3) If F is a sheaf, show that θ_F is an isomorphism.

4) Let E be a topological space, $p\colon E \to X$ a local homeomorphism such that for every $x \in X$, $p^{-1}(x)$ is an abelian group. Define $E \times_X E$ to be the subspace of $E \times E$ consisting of pairs (e, e') with $p(e) = p(e')$. Suppose the two maps $E \times_X E \to E$, $(e, e') \mapsto e + e'$ and $E \to E$, $e \mapsto -e$ are continuous. Let $\mathscr{F} = \mathscr{F}_E$ be the sheaf of sections of $p\colon E \to X$, i.e., for an open subset U of X, $\mathscr{F}(U)$ is the abelian group of continuous maps from $\sigma\colon U \to E$ such that $p \circ \sigma = 1_U$. Show the following.

(a) For $x \in X$, there is a natural isomorphism of abelian groups $\psi_x\colon \mathscr{F}_x \xrightarrow{\sim} p^{-1}(x)$.

(b) There is an isomorphism $\psi\colon \mathscr{E}(\mathscr{F}) \xrightarrow{\sim} E$ such that $p \circ \psi = \pi$.

5) Let X be a topological space, $\mathscr{F}$ a sheaf on X, U an open subset of X, and $\mathfrak{U} = \{U_\alpha\}$ an open cover of U. For every α and β set $U_{\alpha\beta} := U_\alpha \cap U_\beta$. Show that the sequence of abelian groups

$$0 \to \mathscr{F}(U) \xrightarrow{\epsilon} \prod_\alpha \mathscr{F}(U_\alpha) \xrightarrow{d^0} \prod_{\alpha,\beta} \mathscr{F}(U_{\alpha\beta})$$

[1] In other words, s_x is the image of $s \in F(U)$ in the stalk F_x under the natural map $F(U) \to F_x$ arising from the definition of a direct limit.

is exact, where ϵ is the "diagonal" map $s \mapsto (s|_{U_\alpha})_\alpha$ and the map d^0 is defined by $d^0((s_\alpha)_\alpha) = (\sigma_{\alpha\beta})_{\alpha,\beta}$ where $\sigma_{\alpha\beta} = s_\beta|_{U_{\alpha\beta}} - s_\alpha|_{U_{\alpha\beta}}$.

$\mathscr{B}$-sheaves. For the remaining problems consider the following. Let X be a topological space, $\mathscr{B}$ a basis for the topology on X with the extra condition that if B_1 and B_2 are in $\mathscr{B}$ then so is $B_1 \cap B_2$ (e.g. the standard basis for the topology on $\mathrm{Spec}(A)$, where A is a commutative ring). Let F be a $\mathscr{B}$-sheaf (defined in class). For U an open set of X set

$$\mathscr{F}(U) := \ker \left[\prod_{\alpha} F(U_\alpha) \xrightarrow{d^0} \prod_{\alpha,\beta} F(U_{\alpha\beta}) \right] \qquad (*)$$

where (U_α) is an open cover of U with $U_\alpha \in \mathscr{B}$ for every α and d^0 is as in **5**).

6) Show that $\mathscr{F}(U)$ does not depend on the open cover (U_α) of U, i.e. any two covers by members of $\mathscr{B}$ give rise to isomorphic kernels as in $(*)$.

7) Show that the assignment $U \mapsto \mathscr{F}(U)$ gives us a sheaf, which we will denote $\mathscr{F}$.

8) Show that we have an isomorphism of $\mathscr{B}$-sheaves $\mathscr{F}|_{\mathscr{B}} \xrightarrow{\sim} F$.

9) If G is a $\mathscr{B}$-sheaf and $\varphi\colon F \to G$ a map of $\mathscr{B}$-sheaves and if $\mathscr{G}$ is the sheaf on X arising from G via the process outlined in **7**) then show that there is a map $\tilde{\varphi}\colon \mathscr{F} \to \mathscr{G}$ such that the diagram

$$\begin{array}{ccc} \mathscr{F}|_{\mathscr{B}} & \xrightarrow{\sim} & F \\ {\scriptstyle\tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathscr{G}|_{\mathscr{B}} & \xrightarrow{\sim} & G \end{array}$$

commutes, where the horizontal isomorphisms are as in **8**).

10) Show that

$$\mathscr{F}(U) \xrightarrow{\sim} \varprojlim_{B} F(B)$$

where the inverse limit is taken over B such that $B \in \mathscr{B}$ and $B \subset U$.