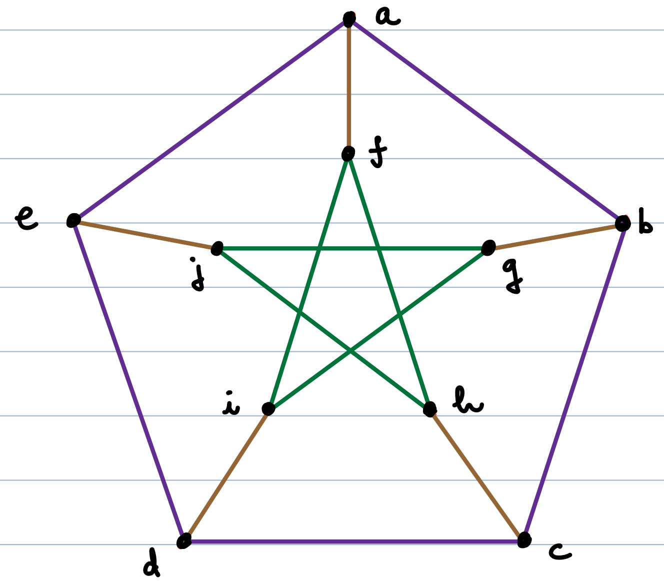
The Petersen graph

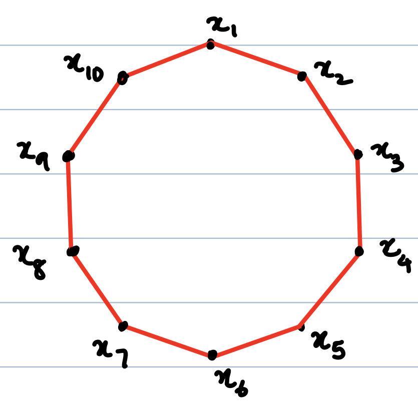
Consider the Petersen graph below



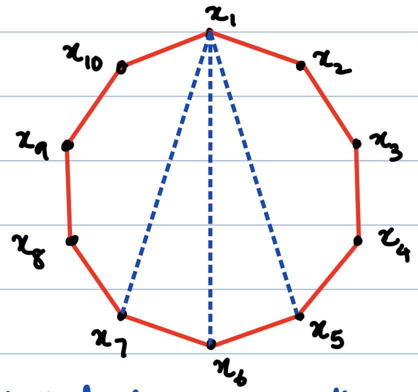
One checks easily that

1. it has 10 vertices and 15 edges.
2. every vertex has degree 3.
3. It is connected
4. It has no cycles of length 3 or 4. All cycles are of length 5 or more.

From these observations it is not hard to see that it is not hamiltonian. To see this let G denote the Petersen graph, and suppose it has hamiltonian cycle $\sigma = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})$. We have to derive a contradiction.

$\sigma =$ 

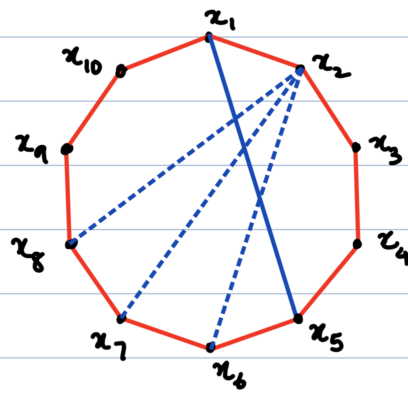
The cycle only accounts for 10 vertices. We need 5 more vertices. Moreover, since each vertex of G has degree 3, we need one edge more for each of the vertices. Let e_i be the edge incident to x_i not traversed by σ. Consider the possibilities for e_1. Since G has no cycles of length 3 or 4, the only possibilities for e_1 are x_1x_5, x_1x_6 and x_1x_7. The same reasoning shows that each e_i has only three possibilities.



The dotted lines are the only possibilities for e_1.

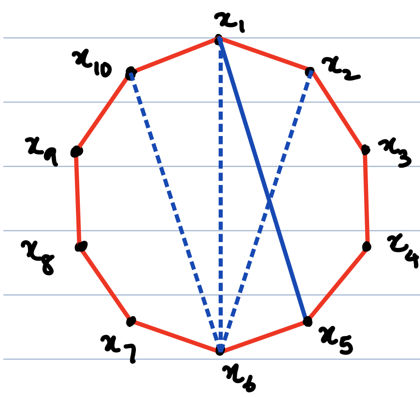
The picture on the left shows the possibilities for e_1. We will show that none of them can occur. Note that if we show e_1 cannot be x_1x_5 then by symmetry e_1 cannot be x_1x_7 either.

I. The case $e_1 = x_1x_5$: There are three possibilities for e_2, as shown by the dotted lines on the picture on the right.



The subcase $e_2 = x_2x_6$: In this case (x_1, x_2, x_6, x_5) is a 4-cycle, which is not possible.

The subcase $e_2 \neq x_2x_6$: We consider e_6.



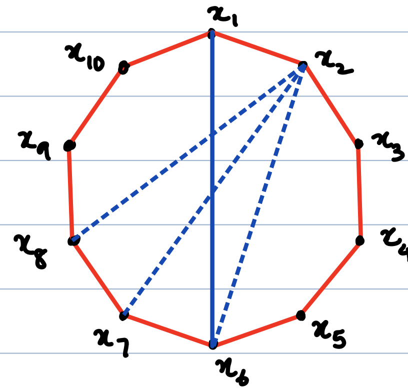
The dotted lines on the picture on the left give the possiblities for e_6. Since $e_2 \neq x_2x_6$, therefore $e_6 \neq x_6x_2$. It cannot be x_6x_1 either, for then $\deg(x_1) \geq 4$, which is not possible. This leaves $e_6 = x_6x_{10}$ as the only possibitity. In this case (x_1, x_{10}, x_6, x_5) is a 4-cycle, which is also not possible.

From our analysis of the two subcases above we conclude that the case $e_1 = x_1x_5$ is not possible.

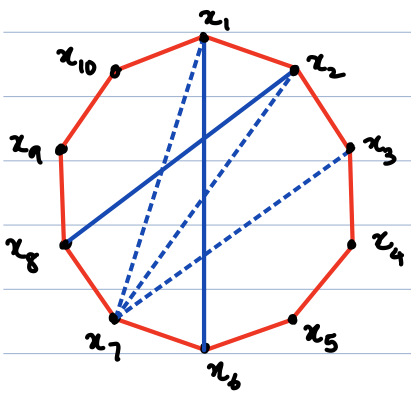
II. The case $e_1 = x_1x_7$: This case is the mirror image of case I, and hence, by symmetry, is not possible.

III. The case $e_1 = x_1x_6$: Once again consider the possibilities for e_2 as shown in the picture below.

If $e_2 = x_2x_6$, then $\deg x_6 \geq 4$, which is not possible. If $e_2 = x_2x_7$, then (x_1, x_6, x_7, x_2) is a 4-cycle in G, which to is not possible. This leaves us with only one subcase, namely $e_2 = x_2x_8$.



So suppose $e_1 = x_1x_6$, and $e_2 = x_2x_8$. There are three possibilities for e_7, and these are shown via dotted lines on the picture on the left.



The possibilities $e_7 = x_7x_1$ and $e_7 = x_7x_2$ are eliminated via degree considerations. If $e_7 = x_7x_3$ then (x_2, x_8, x_7, x_3) is a 4-cycle in G, which is impossible.

Thus none of the possibilities in case III are actual possibilities.

The conclusion is that there is no 10-cycle in G.

It follows that the Petersen graph is not hamiltonian.

The following tree gives the structure of the proof and the various cases and subcases. The reason for the impossibility of a route is given at the bottom of each route of cases, subcases and sub-subcases taken.