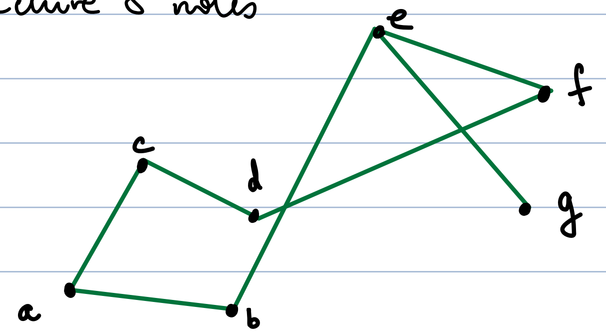
Oct 11, 2022 Lecture 9 MAT 344

From Lecture 8 notes



Two vertices are neighbours if they are adjacent (i.e. there is an edge joining them. The neighbourhood of a vertex is all the vertices which are its neighbours. In the picture above $\{c, f\}$ is the neighbourhood of d and the neighbourhood of g is empty.

Let $G = (V, E)$ be a graph and v a vertex of G.

$$\deg_G(v) := \# \text{ of edges of } G \text{ incident to } v.$$
$$= \# \text{ number of edges in its neighbourhood.}$$

degree of v.

When $G = (V, E)$ and $H = (W, F)$ are graphs, we say H is a subgraph of V if $W \subset V$ and $F \subset E$

Example: In any finite graph, there are at least 2 distinct vertices of the same degree.

Proof: $\deg_G(x) \leq |V| - 1$. So

$$\deg_G : V \longrightarrow [|V| - 1].$$

By the Pigeonhole principle $\deg_G$ is not injective. //

Remark: This is the similar to the example of there being 2 people with the same # of friends in a group of n people.

Theorem: (The Handshake Theorem) Let $G = (V, E)$ be a graph. Then

$$\sum_{v \in V} \deg_G(v) = 2|E|.$$

Proof:

Every edge is incident on two vertices, and contributes twice to the sum of the degrees. //

Corollary: The number of vertices of odd degree is even.

Proof: Let V_o be the set of vertices of odd degree and V_e the set of vertices with even degree. Then

$$2|E| = \sum_{v \in V} \deg(v) = \sum_{v \in V_o} \deg(v) + \sum_{v \in V_e} \deg(v)$$

$$\Rightarrow \sum_{v \in V_o} \deg(v) = 2|E| - \sum_{v \in V_e} \deg(v)$$

even number since $v \in V_e$

even number.

Thus $\sum_{v \in V_o} \deg(v)$ is an even number. But each term in the sum is odd. Since the sum of two odd numbers is even and the sum of an odd number with an even number is odd, it is easy to see that the number of terms in the sum must be even. //

Definitions: Let $G = (V, E)$ be a graph. A *walk* in G is a sequence of vertices $(x_1, x_2, \dots, x_n)$ such that $x_i x_{i+1}$ is an edge for $1 \le i \le n-1$. A *path* is a walk in which all vertices are distinct. A *cycle* is a path $(x_1, x_2, \dots, x_n)$ with $n \geq 3$ such that $x_n x_1$ is an edge.

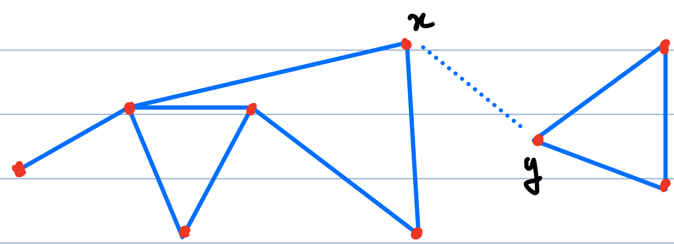
A better definition of a cycle is a walk $(x_1, x_2, \dots, x_n, x_{n+1})$ such that $(x_1, \dots, x_n)$ is a path and $x_{n+1} = x_1$.

The *length* of a walk is the number of edges in it. Thus the length of $(x_1, \dots, x_n)$ is $n-1$.

It is easy to see the following

- length of $P_n = n-1$
- length of $C_n = n$
- A path $(x_1, x_2, \ldots, x_n)$ has length $n-1$
- A cycle $(x_1, x_2, \ldots, x_n, x_1)$ has length n.
- A path of length $n-1$, regarded as a subgraph, is isomorphic to P_n.
- A cycle of length n, regarded as a subgraph, is isomorphic to C_n.

Definition: A graph $G=(V,E)$ is connected if for all $x, y \in V$, $x \neq y$, there is a path starting at x and ending at y.

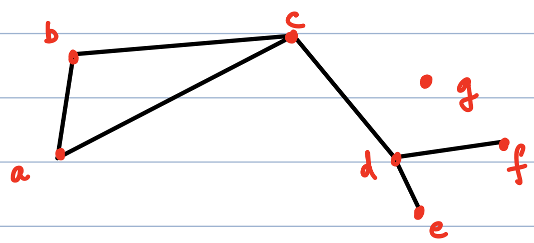


This is not connected. Had the dotted line been an edge then the graph would have been connected.

Definition: A connected component of a graph $G=(V,E)$ is maximal connected subgraph H of G, i.e., if H' is a connected subgraph of G which has H as a subgraph ($H \subseteq H' \subseteq G$) then $H' = H$.

It is clear that G breaks up into connected components, and G is connected if and only if it has only one connected component.

prove this yourself.



There are two connected components in this picture. One is $(\{a,b,c,d,e,f\}, \{ab, ac, bc, cd, de, df\})$ and the other is $(\{g\}, \phi)$.

Eulerian graphs

This requirement may not be standard. Just for this course.

Fix a graph $G=(V,E)$ in the discussion that follows.

Circuits vs cycles: A circuit is a walk $(x_1, x_2, \dots, x_{n+1})$ with $x_{n+1} = x_1$ such that all edges $x_i x_{i+1}$, $i = 1, \dots, n+1$, are distinct. Note that n must be greater than or equal to 3.

A cycle is a circuit $(x_1, x_2, \dots, x_{n+1})$ such that $(x_1, x_2, \dots, x_n)$ is a path, i.e. such that $x_1, x_2, \dots, x_n$ are distinct (and of course, $x_{n+1} = x_1$).

We say that a walk $\sigma = (x_1, x_2, \dots, x_n)$ traverses an edge $e \in E$ if e is one of the edges $x_i x_{i+1}$, $i = 1, \dots, n$.

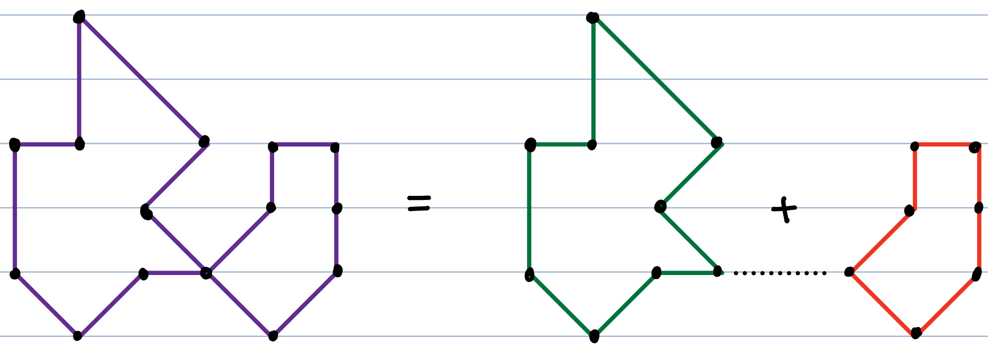
It is clear that if $\sigma = (x_1, \dots, x_n)$ is a path, then regarding σ as the subgraph:

$$H = \left(\{x_1, \dots, x_n\}, \{x_1x_2, x_2x_3, \dots, x_{n-1}x_n\} \right)$$

of G, σ is isomorphic to P_n.

Similarly if $n \geq 3$ and $\sigma = (x_1, x_2, \dots, x_n, x_{n+1})$ is a cycle, then σ is isomorphic to C_n.

It is clear that every circuit breaks up into cycles. I will leave the proof to you. The picture below gives an example of this phenomenon.



σ $\qquad$ σ_1 $\qquad$ σ_2

In the above picture, σ is a circuit and σ_1 and σ_2 are cycles which "add" up to σ.

Definition: A graph $G = (V, E)$ is called eulerian if it is either a graph with only one vertex or it has a circuit which traverses every edge in G exactly once.

Also note that an eulerian graph $G = (V, E)$ is necessarily finite, i.e. the set V is a finite set.

From now on, for the rest of the course, unless otherwise stated, we will assume all graphs are finite.

Theorem: A graph $G = (V, E)$ is eulerian if and only if it is connected and the degree of every vertex in G is an even number.

Proof: The case where G has only one vertex is trivial and there is nothing to prove. From now on we assume that $|V| > 1$ in the proof.

Suppose $G = (V, E)$ is eulerian. Since there is a circuit which traverses every edge, it is clear that any two vertices are connected by a path. In greater detail, suppose $\sigma = (x_1, \dots, x_{n+1})$ is a circuit which traverses every edge. Let $a, b \in V$, $a \neq b$. Then there is an i and a j such that $a = x_i$ and $b = x_j$. We may assume $i < j$. Then the walk $(x_i, x_{i+1}, \dots, x_j)$ is walk which connects a and b. By discarding all the cycles which occur in the walk, we get a path joining a to b. Thus G is connected.

Next suppose $a \in V$, say $a = x_i$. Then the two edges $x_{i-1} x_i$ and $x_i x_{i+1}$ are incident on a (if $i = n+1$, then take $x_{i+1} = x_1$). If a is repeated in the circuit in σ, then for each occurence of a in σ, we have two edges incident to a. Hence the degree of a has to be even.

We will prove the remaining part of the theorem (namely the converse of what we proved) in the next lecture.