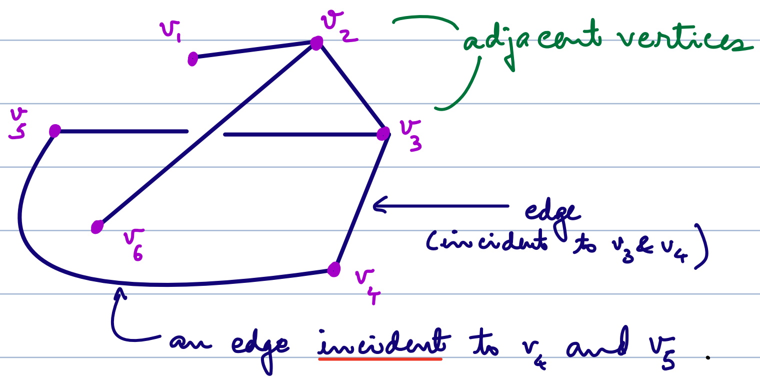
Oct 4, 2022 Lecture 8 MAT 344

5.1 Basic Notation and Terminology for Graphs

A graph looks like this (formal definition after pictures)



Vertices $v_1, v_2, v_3, v_4, v_5, v_6$

$V = \{v_1, v_2, v_3, v_4, v_5, v_6\}$

Edges All the lines and curves in the picture

$E = \{v_1v_2,\ v_2v_3,\ v_2v_6,\ v_3v_4,\ v_3v_5,\ v_4v_5\}$

Graph $= (V, E)$

Here are the more formal definitions.

A graph G is a pair (V, E) where V is a set (usually finite) and E is a set of 2-element subsets of V. Elements of V are called vertices and elements of E are called edges.

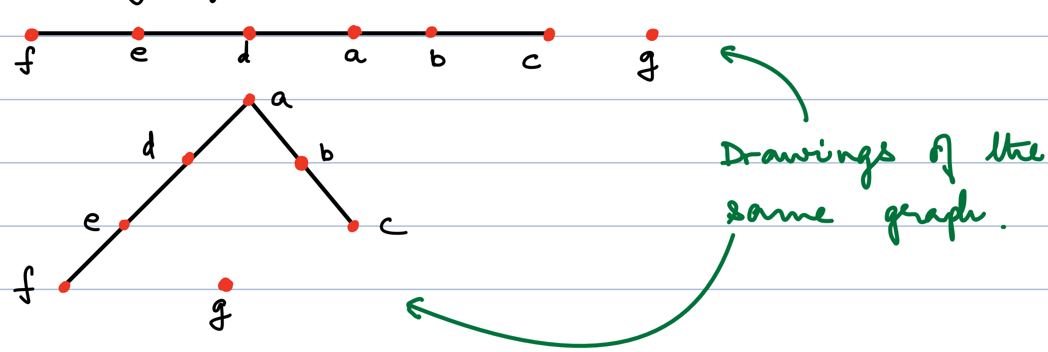
V = vertex set ; E = edge set.

Preferred notation : xy instead of $\{x,y\}$. (Only for graphs!)

Note : $xy = yx$ because $\{x,y\} = \{y,x\}$

If xy is an edge, the vertices x and y are said to be adjacent. The edge xy is said to be incident to x and y

The drawing of a graph is not the same as the graph



Two vertices are neighbours if they are adjacent (i.e. there is an edge joining them. The neighbourhood of a vertex is all the vertices which are its neighbours. In the picture above $\{a, e\}$ is the neighbourhood of d and the neighbourhood of g is empty.

Let $G = (V, E)$ be a graph and v a vertex of G.

$$\deg_G(v) := \# \text{ of edges of } G \text{ incident to } v.$$
$$= \# \text{ number of edges in its neighbourhood.}$$

degree of v.

When $G = (V, E)$ and $H = (W, F)$ are graphs, we say H is a subgraph of V if $W \subset V$ and $F \subset E$

Example: In any finite graph, there are at least 2 distinct vertices of the same degree.

Proof: $\deg_G(x) \leq |V| - 1$. So

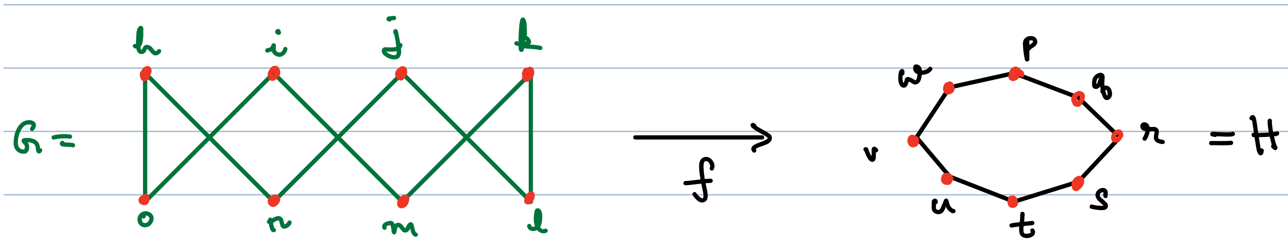
$$\deg_G : V \longrightarrow [|V| - 1].$$

By the Pigeonhole principle $\deg_G$ is not injective. //

Remark: This is the similar to the example of there being 2 people with the same # of friends in a group of n people.

Isomorphisms of graphs

Consider the following two (representations) of graphs



$G = (V, E)$ $\qquad$ $H = (W, F)$

$V = \{h, i, j, k, l, m, n, o\}$, $W = \{p, q, r, s, t, u, v, w\}$

These are essentially the same graphs. The correspondence between the vertices is as follows.

$$
\begin{array}{lll}
h \longleftrightarrow p & \qquad & f(h) = p \\
i \longleftrightarrow v & & f(i) = v \\
j \longleftrightarrow r & & f(j) = r \\
k \longleftrightarrow t & & f(k) = t \\
l \longleftrightarrow s & & f(l) = s \\
m \longleftrightarrow u & & f(m) = u \\
n \longleftrightarrow q & & f(n) = q \\
o \longleftrightarrow w & & f(o) = w
\end{array}
$$

The technical term is G and H are isomorphic.

Definition: We say $G = (V, E)$ and $H = (W, F)$ are isomorphic if there is a bijective map $f : V \longrightarrow W$ such that

$$xy \in E \iff f(x)f(y) \in F.$$

Written as $G \cong H$. (Read as G is isomorphic to H).

Note:

- $G \cong G$
- $G \cong H \Rightarrow H \cong G$
- $G \cong H$ and $H \cong I \Rightarrow G \cong I$.

Isomorphism is an equivalence relation.

Theorem: (The Handshake Theorem) Let $G = (V, E)$ be a graph. Then

$$\sum_{v \in V} \deg_G(v) = 2|E|.$$

Proof:

Every edge is incident on two vertices, and contributes twice to the sum of the degrees. //

Corollary: The number of vertices of odd degree is even.

Proof: Let V_o be the set of vertices of odd degree and V_e the set of vertices with even degree. Then

$$2|E| = \sum_{v \in V} \deg(v) = \sum_{v \in V_o} \deg(v) + \sum_{v \in V_e} \deg(v)$$

$$\Rightarrow \quad \sum_{v \in V_o} \deg(v) = 2|E| - \sum_{v \in V_e} \deg(v)$$

even number since $v \in V_e$

even number.

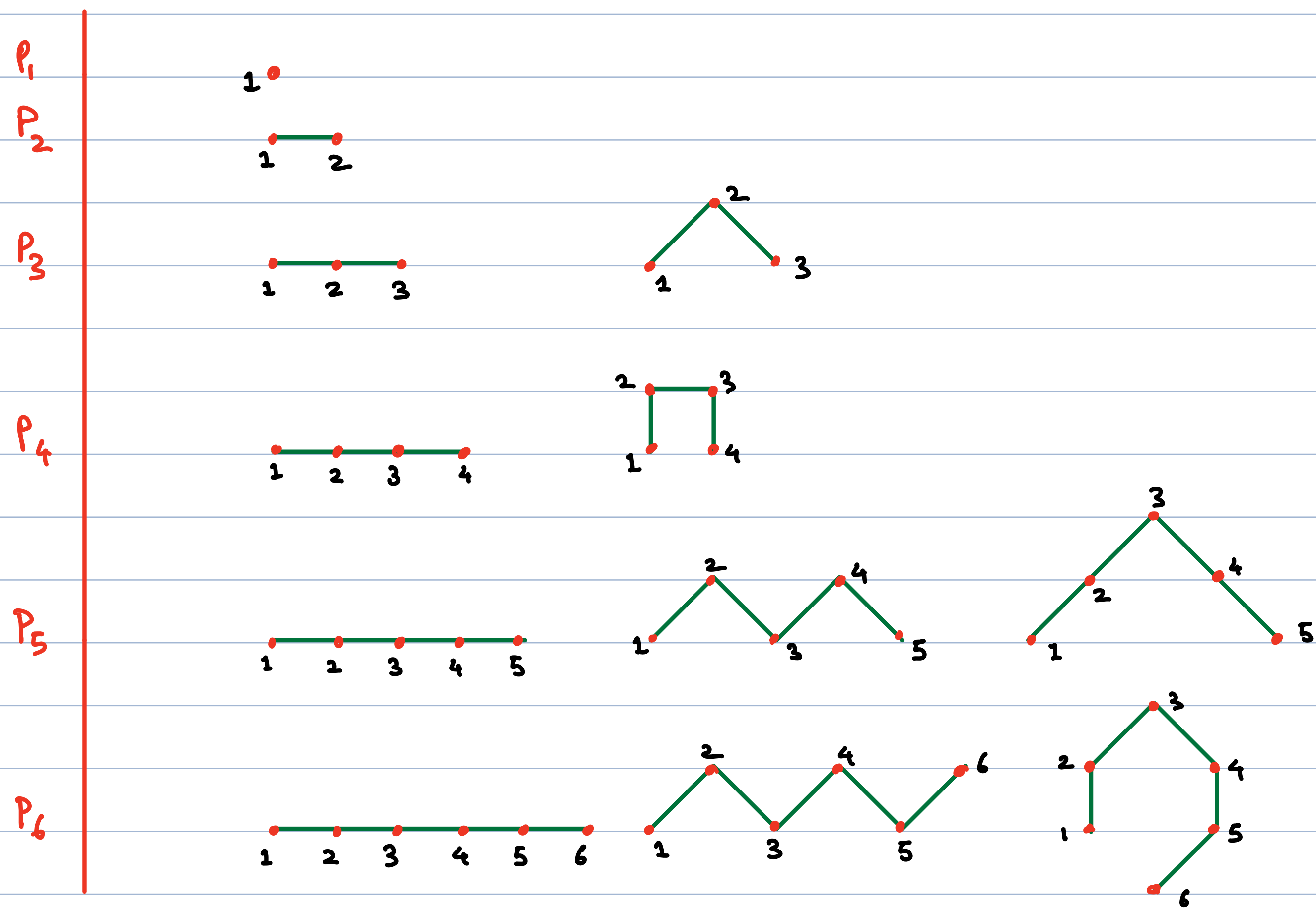
Thus $\sum_{v \in V_o} \deg(v)$ is an even number. But each term in the sum is odd. Since the sum of two odd numbers is even and the sum of an odd number with an even number is odd, it is easy to see that the number of terms in the sum must be even. //

Basic Examples of graphs

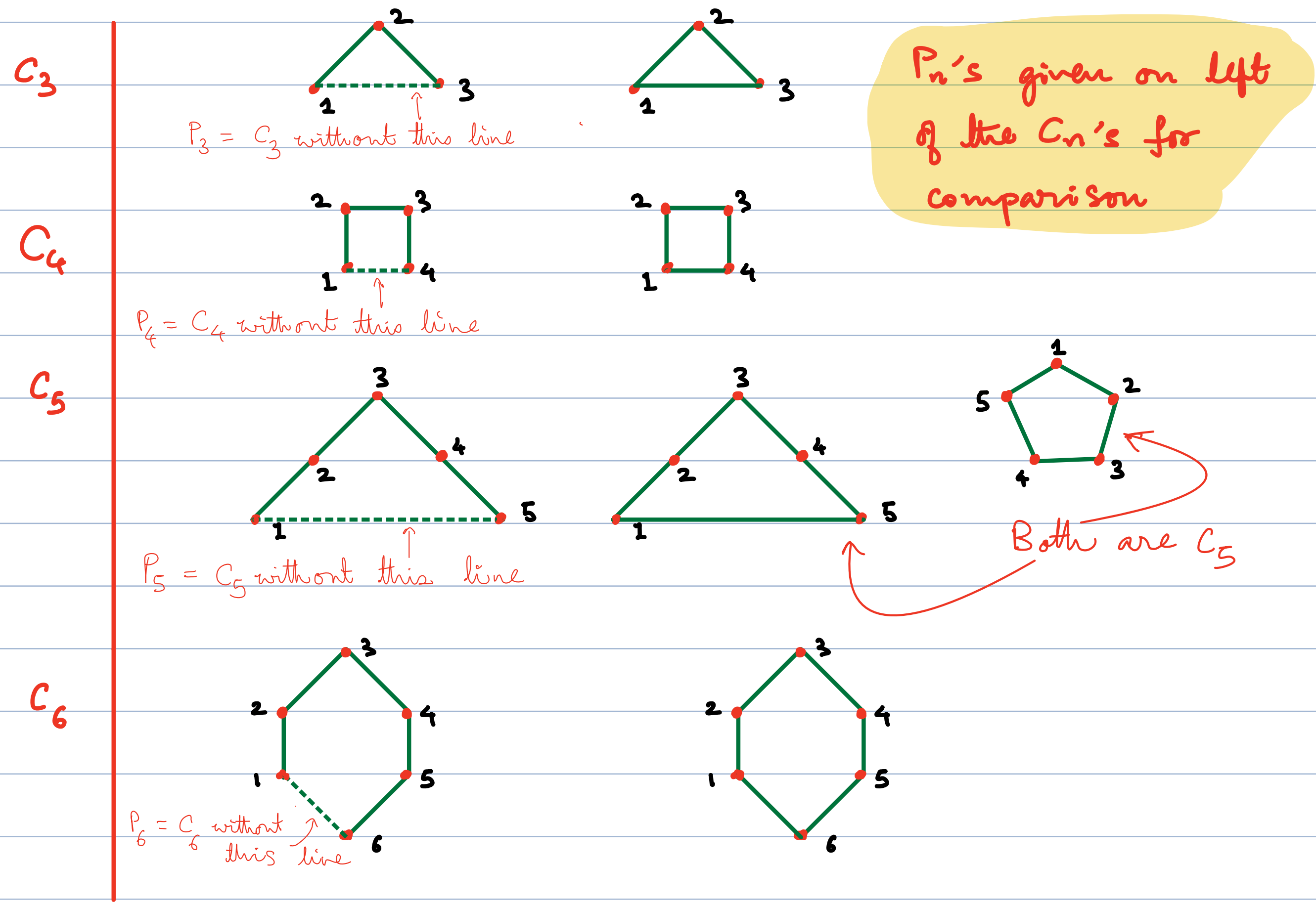
1. $P_n = (V, E)$, $\quad V = \{1, 2, 3, \dots, n\}$

$\qquad E = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}\}$

Here are P_1, P_2, P_3, P_4, , P_5 and P_6, often in more than one version.



2. $C_n = (V, E)$, $V = \{1, 2, \dots, n\}$
$E = \{\{1,2\}, \{2,3\}, \dots, \{n-1, n\}, \{n, 1\}\}$ for $n \geq 3$



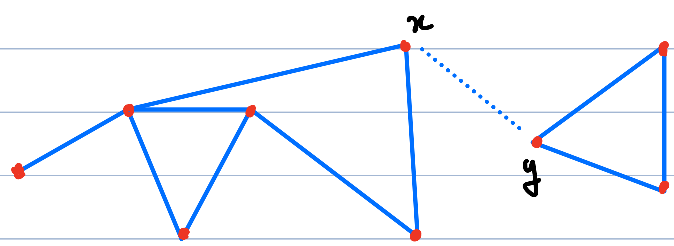
Definitions: Let $G = (V, E)$ be a graph. A walk in G is a sequence of vertices $(x_1, x_2, \dots, x_n)$ such that $x_i x_{i+1}$ is an edge for $1 \leq i \leq n-1$. A path is a walk in which all vertices are distinct.
A cycle is a path $(x_1, x_2, \dots, x_n)$ with $n \geq 3$ such that $x_n x_1$ is an edge.
A better definition of a cycle is a walk $(x_1, x_2, \dots, x_n, x_{n+1})$ such that $(x_1, \dots, x_n)$ is a path and $x_{n+1} = x_1$.
The length of a walk is the number of edges in it.
Thus the length of $(x_1, \dots, x_n)$ is $n-1$.

It is easy to see the following

- length of $P_n = n-1$
- length of $C_n = n$
- A path $(x_1, x_2, \dots, x_n)$ has length $n-1$
- A cycle $(x_1, x_2, \dots, x_n, x_1)$ has length n.

- A path of length $n-1$, regarded as a subgraph, is isomorphic to P_n.
- A cycle of length n, regarded as a subgraph, is isomorphic to C_n.

Definition: A graph $G=(V,E)$ is connected if for all $x,y \in V$, $x \neq y$, there is a path starting at x and ending at y.



This is not connected. Had the dotted line been an edge then the graph would have been connected.