4.1 The Pigeon Hole Principle

The basic idea is:

If we put $n+1$ objects in n boxes, there will be a box with at least 2 objects.

Here is the formal statement.

Theorem: Let X and Y be finite sets such that $|X|>|Y|$. If $f: X \longrightarrow Y$ is a function, then there exist distinct elements $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$, i.e. f is not injective.

Example: Prove that in a group of n people, there are at least 2 persons who know exactly the same number of people in that group, assuming that everyone knows at least person (knowing oneself does not count)

Solution:

A person knows at most $n-1$ people.

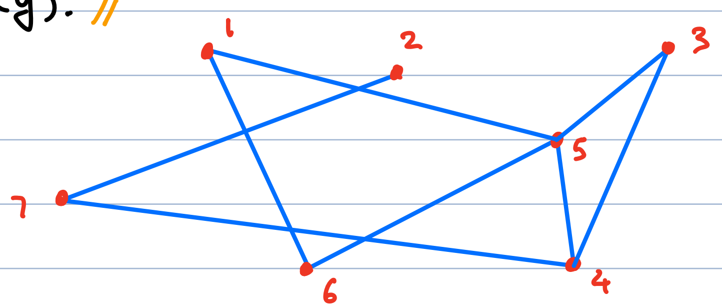
Let X be the set of people in the group and let

$$f: X \longrightarrow [n-1]$$

be defined by

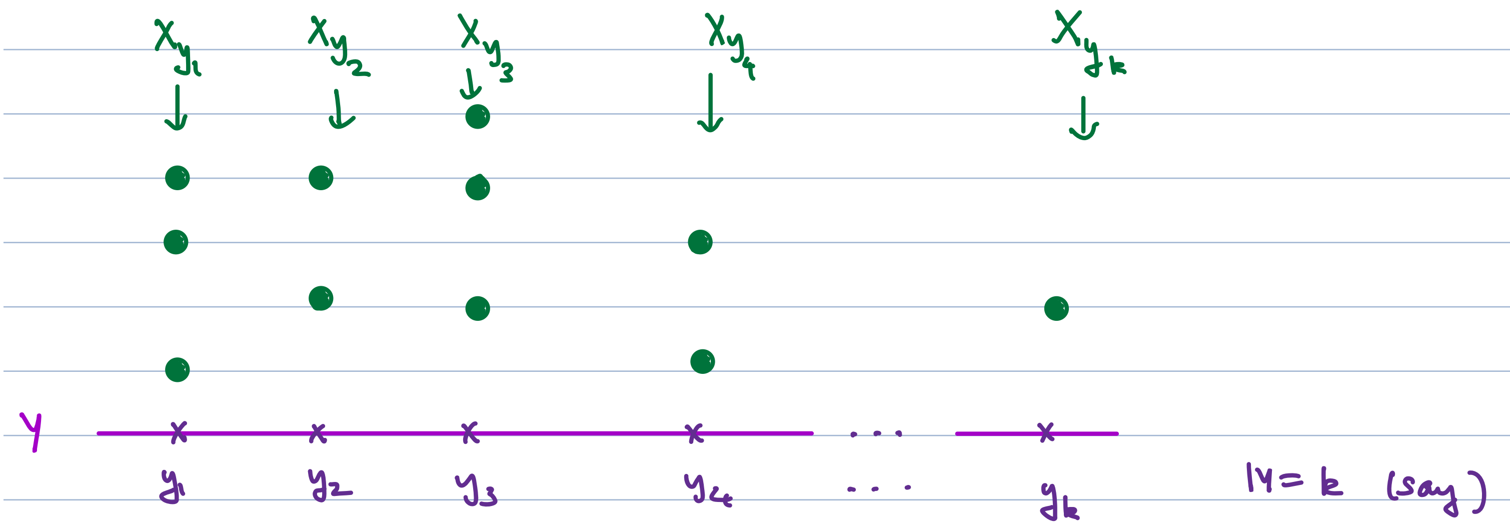
$$f(x) = \#\text{ of people (in the group) } x \text{ knows}$$

Then $|X| = n > n-1 = |[n-1]|$. By the Pigeon Hole Principle, there are distinct people $x, y \in X$ such that $f(x) = f(y)$. //



1 and 7 know 2 persons each. (6 also knows 2 people.)

Proof of the Pigeon Hole principle :



For $y \in Y$, let

$$X_y = \{x \in X \mid f(x) = y\}.$$

Then $$|X| = \sum_{y \in Y} |X_y| .$$

We have to prove that $|X_y| \geq 2$ for some $y \in Y$. Suppose this is not so. Then

$$|X_y| < 2 \qquad \forall\, y \in Y.$$

This means

$$|X_y| \leq 1 \qquad \forall\, y \in Y.$$

Since $|X| = \sum_{y \in Y} |X_y|$, we get

$$|X| = \sum_{y \in Y} |X_y| \leq \sum_{y \in Y} 1 \leq |Y|$$

Thus $|X| < |Y|$, contradicting the hypothesis that $|X| > |Y|$. //

We often write $f^{-1}(y)$ for X_y. The set $f^{-1}(y)$ is often called the inverse image of y under f.

Theorem (Erdös - Szekeres Theorem): Let $m, n \in \mathbb{N}$. Any sequence of $mn+1$ distinct real numbers either has an increasing sequence of $m+1$ terms or a decreasing sequence of $n+1$ terms. (Both are also possible.)

Example: $m=3$, $n=3$, $mn+1 = 9+1 = 10$

$(3, 17, 4, \underline{320}, 5, \underline{33}, \sqrt{2}, \underline{\pi}, 0, \underline{e})$

$(320, 33, \pi, e)$ is a decreasing sequence of length 4.

Actually in this case, we also have an increasing sequence of length 4, namely $(3, 4, 5, 33)$.

Proof:

Let $\sigma = (x_1, x_2, x_3, \dots, x_{mn+1})$ be a finite sequence of $mn+1$ distinct terms. For all $i \in [mn+1]$, let

a_i = maximum number of terms in an increasing subsequence of σ with x_i the first term

b_i = maximum number of terms in a decreasing subsequence of σ with x_i the last term.

We have to show that at least one of the following happens:

- $a_i \geq m+1$ for some i;

or

- $b_i \geq n+1$ for some i

Suppose not. Then $a_i \leq m$ and $b_i \leq n$ for all i.

Let $X = [mn+1] = \{1, 2, \dots, mn+1\}$ and

$$Y = \{(a,b) \mid a, b \in \mathbb{N}, \ 1 \leq a \leq m, \ 1 \leq b \leq n\} = [m] \times [n].$$

Let $f : X \longrightarrow Y$ be the map

$$f(i) = (a_i, b_i) \qquad i = 1, \dots, mn+1.$$

Since $|X| = mn+1 > mn = |Y|$, by the Pigeon Hole principle there exist $i, j \in [mn+1]$ with $i < j$ such that $f(i) = f(j)$, i.e.

$$(a_i, b_i) = (a_j, b_j).$$

There are two possibilities.

1. $x_i < x_j$ and 2. $x_i > x_j$.

If $x_i < x_j$, pick a subsequence $(x_{i_1}, x_{i_2}, \dots, x_{i_{a_j}})$ of σ with $x_{i_1} = x_j$, which is increasing. This is possible because of the definition of a_j. Then the subsequence $(x_i, \underset{x_j}{\underset{\shortparallel}{x_{i_1}}}, \dots, x_{i_{a_j}})$ is increasing and has $a_j + 1 = a_i + 1$ terms (we are using the fact that $(a_i, b_i) = (a_j, b_j)$ whence $a_i = a_j$ and $b_i = b_j$). This is not possible by the definition of a_i.

If on the other hand $x_i > x_j$, we can find a decreasing of σ of length b_i whose last term is x_i. Adding the term x_j at the end of the subsequence, we get a decreasing subsequence of σ of length $b_i + 1 = b_j + 1$ whose last term is x_j. This contradicts the definition of b_j.

We are therefore done. //

Theorem (The Generalized Pigeon Hole principle): Let X and Y be finite sets with

$$|X| > (m-1)|Y|$$

for some $m \in \mathbb{N}$. If $f : X \longrightarrow Y$ is a function, then there exist at least m distinct elements $x_1, x_2, x_3, \dots, x_m$ of X such that $f(x_1) = f(x_2) = \dots = f(x_m)$.

Proof:

As before, for $y \in Y$, let X_y be the inverse image of y under f, i.e.

$$X_y = \{x \in X \mid f(x) = y\}.$$

We have to show that $|X_y| \geq m$ for some $y \in Y$. Suppose this is not so. This means

$$|X_y| < m \qquad \text{for all } y \in Y$$

or, equivalently,

$$|X_y| \leq m-1 \qquad \text{for all } y \in Y.$$

Summing over $y \in Y$, we get

$$|X| = \sum_{y \in Y} |X_y| \leq \sum_{y \in Y} (m-1) = |Y| \cdot (m-1),$$

i.e. $|X| \leq |Y| \cdot (m-1)$. This contradicts the hypothesis of the theorem. //

Example (Ramsey's theorem): In a group of 6 people, there is either a set of 3 people who all know each other, or a set of 3 people none of who know each other.

Proof:

Let the people be x_1, x_2, x_3, x_4, x_5 and x_6.

Define a map

Note that this starts with x_2 and not x_1.

$$f : \{x_2, x_3, x_4, x_5, x_6\} \longrightarrow \{0, 1\}$$

by the rule

$$f(x_i) = \begin{cases} 0 & \text{if } x_1 \text{ and } x_i \text{ do not know each other} \\ 1 & \text{if } x_1 \text{ and } x_i \text{ know each other} \end{cases}$$

Now

$$|\{x_2, x_3, x_4, x_5, x_6\}| > (3-1) \cdot |\{0, 1\}| \text{ since } 5 > (2)(2) = 4.$$

The Generalized Pigeon Hole principle applies and we get a subset $\{x_{i_1}, x_{i_2}, x_{i_3}\}$ of $\{x_2, x_3, x_4, x_5, x_6\}$ such that

$$f(x_{i_1}) = f(x_{i_2}) = f(x_{i_3})$$

There are two cases.

Case 1: $f(x_{i_1}) = f(x_{i_2}) = f(x_{i_3}) = 1$

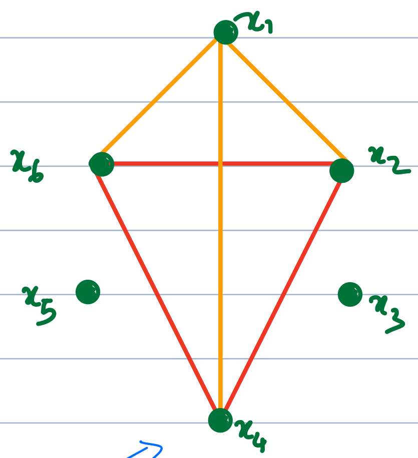
Case 2: $f(x_{i_1}) = f(x_{i_2}) = f(x_{i_3}) = 0$.

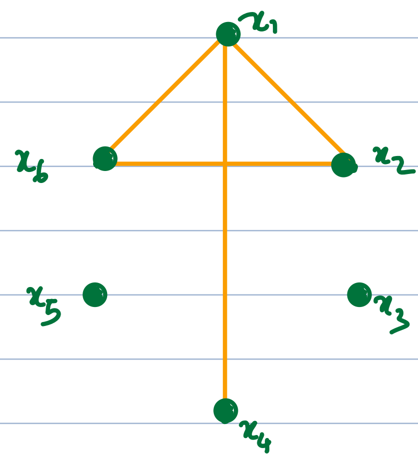
Consider Case 1. Either none of $\{x_{i_1}, x_{i_2}, x_{i_3}\}$ know each other, or at least two of them know each other. In the first case we are done. In the second case, suppose x_{i_1} and x_{i_2} know each other. Then $\{x_1, x_{i_1}, x_{i_2}\}$ is a subset of 3 persons all of who know each other. So in the event we are in Case 1, we are done. The picture below may help.

— = friends
— = not friends

Picture for Case 1

x_2, x_4, x_6 friends with x_1





Subcase: None of x_2, x_4, x_6 friends with each other. The subset $\{x_2, x_4, x_6\}$ works.

Subcase: x_2 and x_6 friends with each other. The subset $\{x_1, x_2, x_6\}$ works.

Now suppose we are in Case 2. In this case either all three of $x_{i_1}, x_{i_2}, x_{i_3}$ know each other, or there are two of them (say x_{i_1} and x_{i_2}) who do not know each other. In the former case, we are done. In the latter case, none of x_1, x_{i_1}, x_{i_2} know each other. So we are done in this case too. //