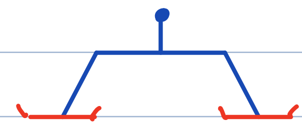
Example: You are given 3^n coins, all identical except for one which is heavier. Using a balance, prove that you can find the heavy coin in n weighings.

Solution

If $n=1$, we have only 3 coins. Pick any two and compare them on the balance. If the two coins have the same weight, then the one left out is the heaviest. Otherwise the balance tells us which is heavier. So one weighing is enough.



We now know how to work this when $n=1$. Now let us work this out for $n=2$. We have 9 coins. We can make three piles out of them of three coins each. Pick any two piles and compare them on the balance. If they balance, the pile left out is heaviest. If not, the balance tells us which one is heavier. In either case, after one weighing we have a pile of three coins, identical, except for one coin which is heavier. We have already seen (from the $n=1$ case) that one more weighing and we can identify the heavy coin.

How would you do a pile of $3^3=27$ coins? The idea is now clear. Split it into 3 piles of 9 coins each. One weighing tells you which pile of $9=3^2$ coins has the heavy coin. The previous case tells you that 2 further weighings and you have identified the heavy coin. In total, three weighings are enough when $n=3$.

In general, if $n\geq 2$, and you have worked out how to identify the heavy coin for a pile of 3^{n-1} coins in $n-1$ weighings, then for a pile of 3^n coins, you should break this pile into 3 piles of 3^{n-1} coins each. In one weighing you can work out which pile of 3^{n-1} coins has the heavy coin. And since you have worked out the $n-1$ case, in another $n-1$ weighings you are done.

Example: Find an upper bound for the sequence

$\sqrt{2}, \sqrt{2+\sqrt{2}}, \sqrt{2+\sqrt{2+\sqrt{2}}}, \ldots$

Solution:

Since $2 \le 4$, $\sqrt{2} \le \sqrt{4} = 2$.

What about $\sqrt{2+\sqrt{2}}$?

We have $2+\sqrt{2} \le 2+2 = 4$, and hence

$$\sqrt{2+\sqrt{2}} \le \sqrt{4} = 2$$

What about $\sqrt{2+\sqrt{2+\sqrt{2}}}$?

We have

$$2+\sqrt{2+\sqrt{2}} \le 2+2 \le 4$$

and hence

$$\sqrt{2+\sqrt{2+\sqrt{2}}} \le \sqrt{4} = 2.$$

If we continue in this way, it looks very likely that 2 is an upper bound for the sequence. Here is the basic idea. Let s_n be the n^{th} term of the sequence. Then

$$s_{n+1} = \sqrt{2+s_n}, \quad n \ge 1.$$

In our calculations above, we first showed $s_1 \le 2$. Then used the fact that $s_1 \le 2$ to show that $s_2 \le 2$. And then we used the fact that $s_2 \le 2$ to show that $s_3 \le 2$.

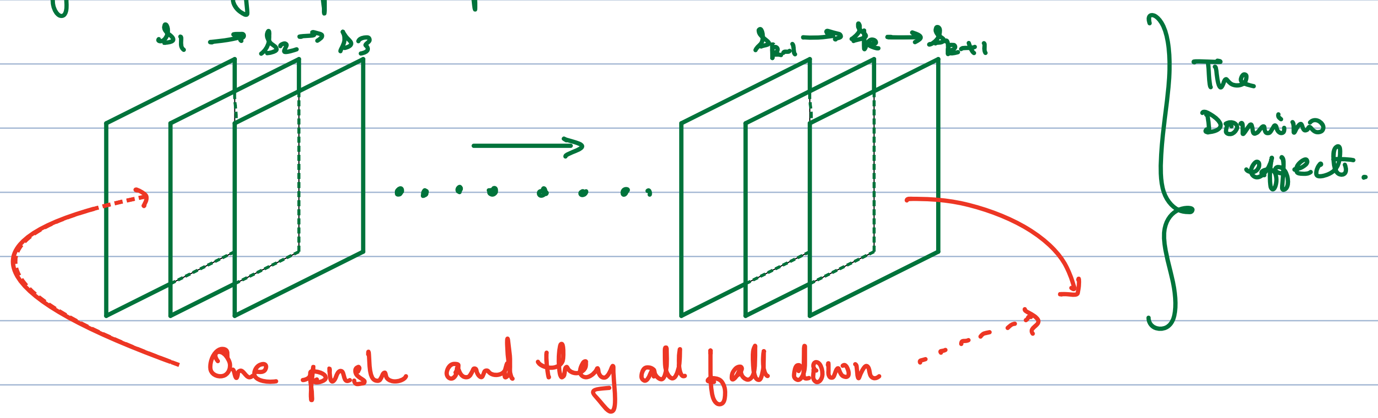
In general, suppose we have shown that $s_k \le 2$.

Since $s_{k+1} = \sqrt{2+s_k}$, we see that

$$s_{k+1}^2 = 2+s_k \le 2+2 = 4.$$

Hence $s_{k+1} \le 2.$ //

The examples above seem to fall under the following organising principle.



The principle of mathematical induction

Let $S_1, S_2, S_3, \ldots$ be a sequence of statements (e.g. equations) such that

(base case) $\quad S_1$ is true

(Inductive step) $\quad S_k \Rightarrow S_{k+1}$ for all $k \in \mathbb{N}$.

Then, S_n is true for all $n \in \mathbb{N}$.

The picture of the dominoes above should help with understanding the principle.

Example: Show that $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$, $n \in \mathbb{N}$.

Proof by induction: Let S_n be the assertion above.

The base case, when $n=1$, is true because

$$\sum_{i=1}^{1} i = 1 = \frac{1(1+1)}{1}$$

S_1 is true

Suppose for some $k \geq 1$, the statement is true for $n=k$. Let us show that S_{k+1} is true.

Suppose S_k is true for some $k \geq 1$

$$\sum_{i=1}^{k+1} i = \left(\sum_{i=1}^{k} i\right) + (k+1)$$

$$= \frac{k(k+1)}{2} + (k+1) \qquad \text{(since } S_k \text{ is true)}$$

$$= \frac{k(k+1) + 2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

This shows that S_{k+2} is true, i.e.

$$S_k \Rightarrow S_{k+1}.$$

By the principle of induction S_n is true $\forall\, n \in \mathbb{N}$ q.e.d.

Example: Show that $1+3+5+\dots+(2n-1) = n^2$, $n \in \mathbb{N}$.

Solution:

S_n : $1+3+5+\dots+(2n-1) = n^2$

Have to show S_n is true for all $n \in \mathbb{N}$.

Set $n=1$. The L.S. of $S_n = 1$ & the R.S. of $S_n = 1$. So S_1 is true (base case)

Suppose S_k is true for some $k \geq 1$.

L.S. of S_{k+1} is

$$1+3+5+\dots+(2k-1)+(2k+1)$$
$$= 1+3+5+\dots+(2k-1)+(2k+1)$$
$$= k^2 + (2k+1) \quad \text{(since } S_k \text{ is true).}$$
$$= (k+1)^2$$
$$= \text{R.S. of } S_{k+1}.$$

Thus

$$S_k \Rightarrow S_{k+1} \qquad k \in \mathbb{N}.$$

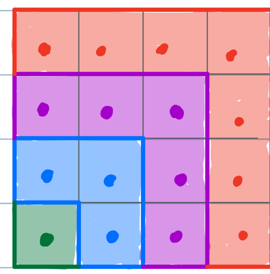
By induction, S_n is true for all n. //

Pictorial proof

$1+3+5+7$

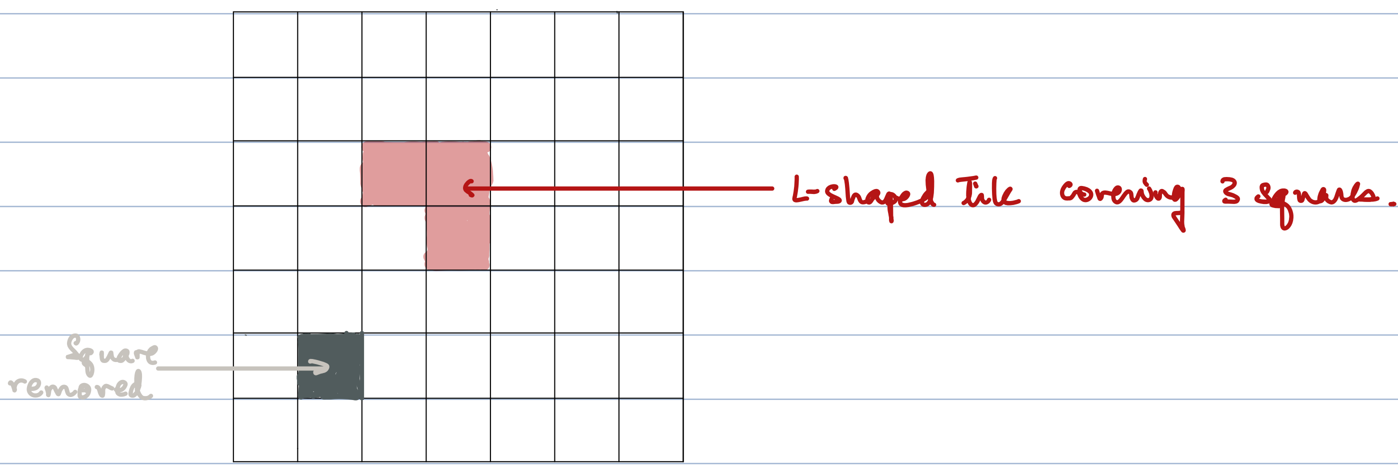
$= 16.$

$n=4$



The general case can also be proved by drawing an $n \times n$ square and using the same trick.

Example: Let n be a positive integer. Show that any $2^n \times 2^n$ chessboard with one square removed can be tiled using using L-shaped pieces, where these pieces cover three squares at a time.



Write out the solution yourself. The solution was discussed in class. Hint: If you know how to tile a $2^{n-1} \times 2^{n-1}$ chessboard with a square removed, can you use that knowledge to tile a

$2^n \times 2^n$ chessboard with a square removed?

Sometimes induction is NOT strong enough.

Example: Define $a_1, a_2, a_3, \ldots$ by the recursive relations

$$a_1 = 3, \quad a_2 = 5$$

$$a_n = 2a_{n-1} - a_{n-2}, \qquad n \geqslant 3.$$

Let us compute a_3, a_4, a_5.

$$a_3 = 2a_2 - a_1 = 2(5) - 3 = 7$$

$$a_4 = 2a_3 - a_2 = 2(7) - 5 = 9$$

$$a_5 = 2a_4 - a_3 = 2(9) - 7 = 11$$

$a_1 = 3$
$a_2 = 5$
$a_3 = 7$
$a_4 = 9$
$a_5 = 11$

Pattern suggest that $a_n = 2n + 1$

What is a_{100}? Chances are it is $2(100)+1 = 201$.

Claim: $a_n = 2n+1$.

Will induction work?

The statement is certainly true for $n=1$.

Suppose it is true for $n=k$, i.e. suppose we know

$$a_k = 2k+1$$

for some $k \geqslant 2$. (We know it for $k=1$ & 2 by inspection)

$$a_{k+1} = 2a_k - a_{k-1}$$

$$= 2(2k+1) - a_{k-1}$$

$$= 4k + 2 - a_{k-1}$$

What do we do with this?

We are stuck. So the principle of induction (in the form we have given it) does not quite help here.

What if we assumed that for $k \geq 2$, the statement is true $n=k$ AND for $n=k-1$?

Let us do that. We have already seen that

$$a_{k+1} = 4k+2 - a_{k-1}.$$

Since the statement is assumed true for $n=k-1$ also, the above gives

$$\begin{aligned} a_{k+1} &= 4k+2 - (2(k-1)+1) \\ &= 4k+2 - (2k-1) \\ &= 2k+3 \\ &= 2(k+1)+1. \end{aligned}$$

Which means

$$S_k + S_{k-1} \Rightarrow S_{k+1}. \qquad (*)$$

Since the statement is true for $n=1$ & 2, by $(*)$ it is true for $n=3$. Now the statement is true for $n=2$ and $n=3$. So it must be true for $n=4$. Continuing this way we see that S_n is true for all $n \in \mathbb{N}$.

We clearly need to strengthen our principle of induction in light of the above example.

The Strong Principle of Mathematical Induction

Let $S_1, S_2, \ldots,$ be a sequence of statements such that:

(Base case) S_1 is true,

and for all $k \in \mathbb{N}$

(Inductive step) For all $k \in \mathbb{N}$, S_{k+1} is true whenever $S_1, S_2, S_3, \ldots, S_k$ are true. Then, S_n is true for all $n \in \mathbb{N}$.

(usual induction)

Example: Let

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \psi = \frac{1-\sqrt{5}}{2}.$$

(For the record, ϕ is the "golden ratio", and ϕ and ψ are the roots of $x^2 - x - 1$.) Show that

$$F_n = \frac{\phi^n - \psi^n}{\phi - \psi}, \qquad n \in \mathbb{N}.$$

Fibonacci numbers
$F_n = F_{n-1} + F_{n-2}$, $F_1 = 1$, $F_2 = 1$.

Proof: We will have to use the strong principle in this case too (try using the usual version of induction - you will fail).

The base case is clear.

(Base case) S_1 is clearly true. The left side is F_1 which equals 1, and the right side is $\frac{\phi - \psi}{\phi - \psi}$ which also equals 1.

What about $n = 2$? Note $F_2 = 1$. On the other hand

$$\frac{\phi^2 - \psi^2}{\phi - \psi} = \frac{(\phi - \psi)(\phi + \psi)}{\phi - \psi}$$

$$= \phi + \psi$$

$$= \frac{1+\sqrt{5}}{2} + \frac{1-\sqrt{5}}{2}$$

$$= 1$$

$$= F_2.$$

So S_2 is also true.

(Induction step) Suppose the formula holds for $n = 1, 2, 3, \dots, k$ for some $k \geq 2$, i.e. $S_1, \dots, S_k$ are true for some $k \geq 2$. Then

$$F_{k+1} = F_k + F_{k-1} \qquad \text{(we are using } k \geq 2 \text{ here)}$$

$$= \frac{\phi^k - \psi^k}{\phi - \psi} + \frac{\phi^{k-1} - \psi^{k-1}}{\phi - \psi}$$

$$= \frac{\phi^{k-1}(1+\phi) - \psi^{k-1}(1+\psi)}{\phi - \psi}$$

$$= \frac{\phi^{k-1}(\phi^2) - \psi^{k-1}(\psi^2)}{\phi - \psi} \quad \left(\text{since } \phi \text{ and } \psi \text{ satisfy } x^2 - x - 1 = 0\right)$$

$$= \frac{\phi^{k+1} - \psi^{k+1}}{\phi - \psi}$$

This means S_{k+1} is also true.

q.e.d.