Chapter 3 - Induction

Recursion

Example: Define $n!$ (for $n \in \mathbb{N}$) by

$$\left.\begin{aligned} 1! &= 1 \\ n! &= n \cdot (n-1)! \quad \text{for } n > 1. \end{aligned}\right\} \text{Recursive definition.}$$

Example: Define $F_1, F_2, F_3, \ldots$ (the Fibonacci numbers) by

$$\left.\begin{aligned} &F_1 = 1, \quad F_2 = 1 \\ &F_n = F_{n-1} + F_{n-2} \quad \text{for } n > 2. \end{aligned}\right\} \text{Recursive defn.}$$

F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8
1	1	2	3	5	8	13	21

Example: The relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \qquad (0 \le k \le n)$$

we proved in Lecture 2 is a recurrence relation.

The equations

$$\binom{n}{0} = 1, \quad \binom{n}{n} = 1, \quad \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

determine $\binom{n}{k}$ for all n and $0 \le k \le n$.

```
                1                    n=0
            1       1                n=1
        1       2       1            n=2      PASCAL'S
    1       3       3       1        n=3      TRIANGLE
  1     4       6       4       1    n=4
1    5     10      10      5     1   n=5
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Example (Catalan numbers): The relation

$$\left.\begin{aligned} T_0 &= 1 \\ T_{n+1} &= \sum_{k=0}^{n} T_k T_{n-k} \end{aligned}\right\}$$

is a recurrence relation.

n=0 $\quad T_1 = \sum_{k=0}^{0} T_k T_{0-k} = T_0 T_0 = 1.$

n=1 $\quad T_2 = \sum_{k=0}^{1} T_k T_{1-k} = T_0 T_1 + T_1 T_0 = 1+1=2.$

n=2 $\quad T_3 = \sum_{k=0}^{2} T_k T_{2-k} = T_0 T_2 + T_1 T_1 + T_2 T_0 = 2+1+2=5.$

$\vdots$

See note on Quercus on diagonal lattice paths.

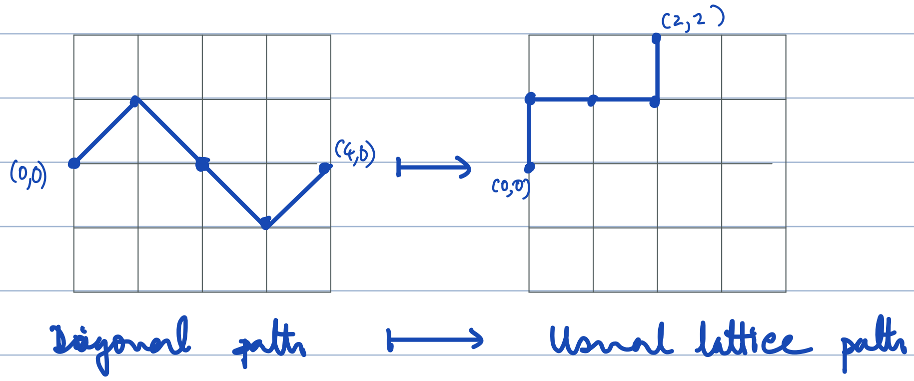
Lattice paths help solve the problem. Let us use diagonal paths, for no particular reason except for variation. Recall that diagonal lattice paths composed of moves which are either northeast moves (corresponding to vertical moves in the usual lattice paths) and south east moves (corresponding to horizontal moves in the usual lattice paths). One can move back and forth between diagonal and usual lattice paths by using the transformations

$$\begin{bmatrix} p \\ q \end{bmatrix} \longmapsto \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} p \\ q \end{bmatrix} \qquad \text{(diagonal to usual)}$$

$$\begin{bmatrix} m \\ n \end{bmatrix} \longmapsto \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} m \\ n \end{bmatrix} \qquad \text{(usual to diagonal).}$$

It is easy to see that points of the (m,m) in usual paths correspond to points of the form $(2m, 0)$ in diagonal paths and vice-versa.

Catalan or Dyke paths in the diagonal world are defined to be paths from $(0,0)$ to $(2n, 0)$ which never dip below the x-axis.



Let $(2l, 0)$ be the first time after $(0,0)$ that a Catalan path hits the x-axis. This means that from $(1,1)$ to $(2l-1, 1)$ our Catalan path lies above the line $y=1$ (the green horizontal line). This is essentially a Catalan path from $(0,0)$ to $(2l-2, 0)$. (Shift the origin to $(1,1)$.) Here $1 \le l \le n$. The path from $(2l, 0)$ to $(2n, 0)$ is essentially a Catalan path from $(0,0)$ to $(2n-2l, 0)$.

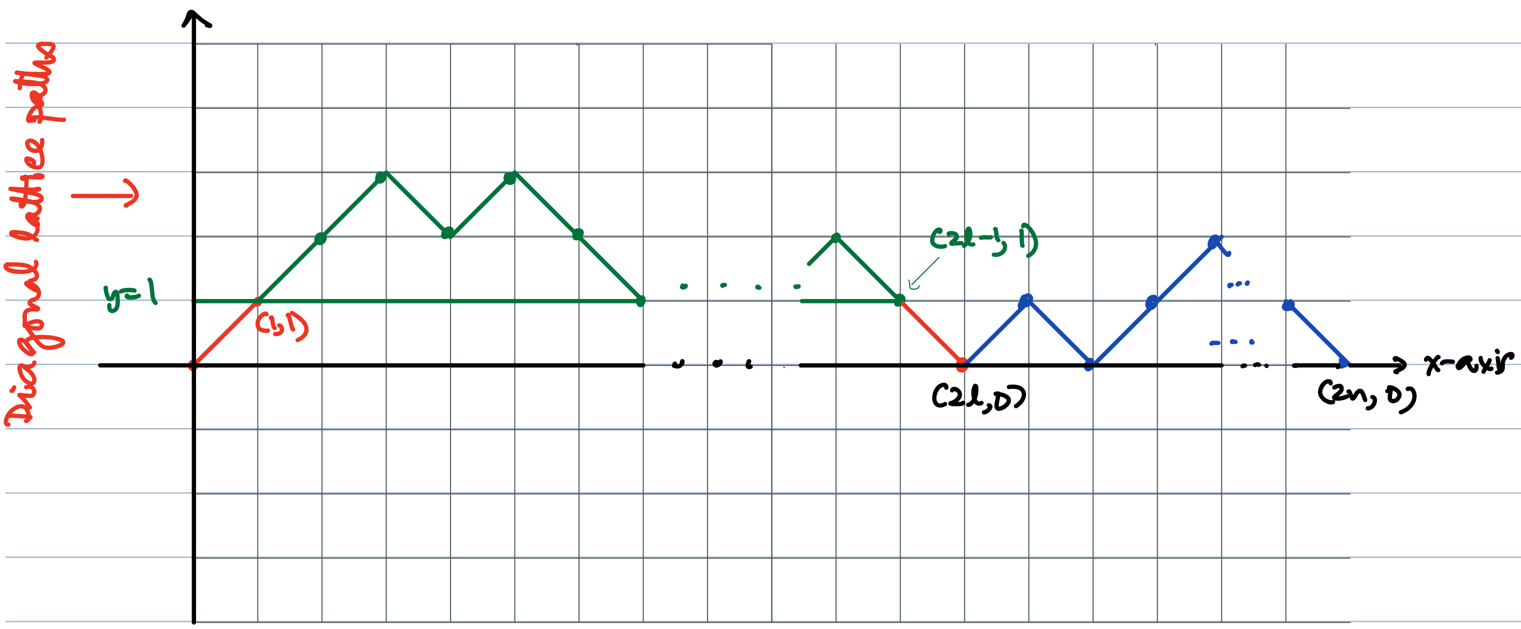


FIGURE: A Catalan (or a Dyck) path from $(0,0)$ to $(2n, 0)$. The first time it hits the x-axis after $(0,0)$ is at $(2l, 0)$.

It follows that for $1 \le l \le n$, the number of Catalan paths from $(0,0)$ to $(2n, 0)$ which hit the x-axis for the first time $(2l, 0)$ (after $(0,0)$) is is $C_{l-1} \cdot C_{n-l}$ where $C_i = \frac{1}{i+1}\binom{2i}{i}$.

We therefore have

$$C_n = \sum_{l=1}^{n} C_{l-1} \cdot C_{n-l}$$

Setting $k = l-1$ in the above, so that $l = k+1$, and $0 \le k \le n-1$, we get

$$C_n = \sum_{k=0}^{n-1} C_k \cdot C_{n-k-1}$$

and hence

$$C_{n+1} = \sum_{k=0}^{n} C_k \cdot C_{n-k}, \qquad C_0 = 1.$$

Thus the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ satisfy our original recurrence relation

$$T_{n+1} = \sum_{k=0}^{n} T_k T_{n-k}, \qquad T_0 = 1. \qquad \text{———} (*)$$

Since $(*)$ has a unique solution, we have proved the the only solution of $(*)$ is

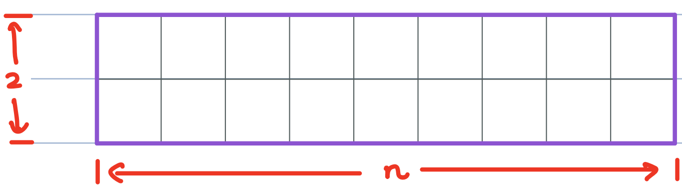
$$T_n = \frac{1}{n+1}\binom{2n}{n}, \quad n \geq 0.$$

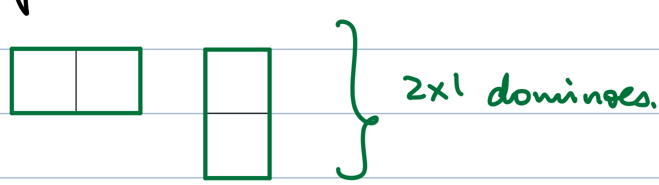
This gives us a recursive definition of Catalan numbers, namely the set of numbers $C_0, C_1, C_2, \ldots, C_n, \ldots$ satisfying the recurrence relation

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}, \quad C_0 = 1, \quad n \geq 0.$$

Remark: If we specify our initial values of our variables in our recurrence relation, then we have unique solutions to our recurrence relation.

Example: Consider a box of size $2 \times n$. In how many ways can you tile this box with 2×1 dominoes?

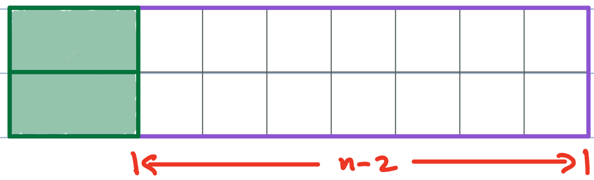




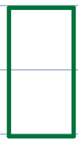
Solution: Let D_n be this number. There are two cases

(1) 

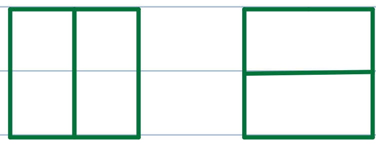
We have D_{n-1} possibilities for the rest.

(2) 

We have D_{n-2} possibilities for the rest.

$D_1 = 1$ 

Only one way to tile when $n=1$.

$D_2 = 2$ 

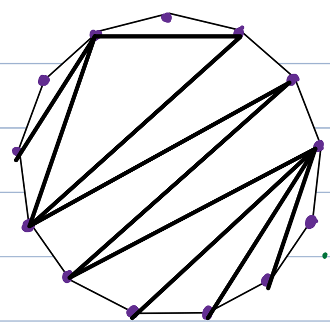
Only two ways to tile when $n=2$.

Thus

$$D_n = D_{n-1} + D_{n-2}, \quad n \geq 2 \qquad F_{n+1} = F_n + F_{n-1}, \quad n \geq 1.$$
$$D_1 = 1, \quad D_2 = 2 \qquad F_2 = 1, \quad F_3 = 2$$

So $D_n = F_{n+1}$, since D_n and F_{n+1} satisfy the same recursive definition.

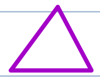
Example: In how many ways can you triangulate a regular polygon with m sides $(m \geq 3)$?



For simplicity, let $m = n+2$. Then $n \geq 1$.

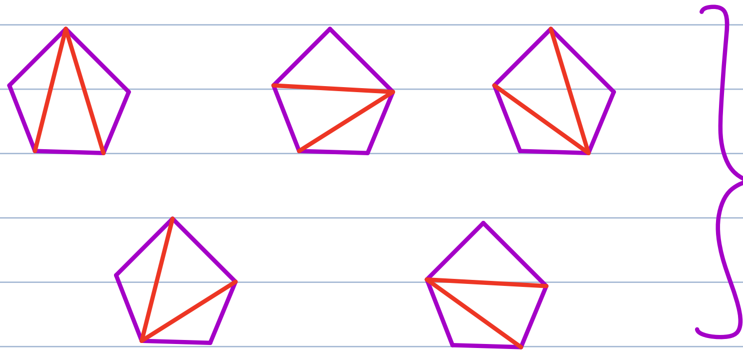
Let $G_n = \#$ of ways to triangulate an $(n+2)$-gon.

To get a feeling for the problem, let us compute a few of the G_n's.

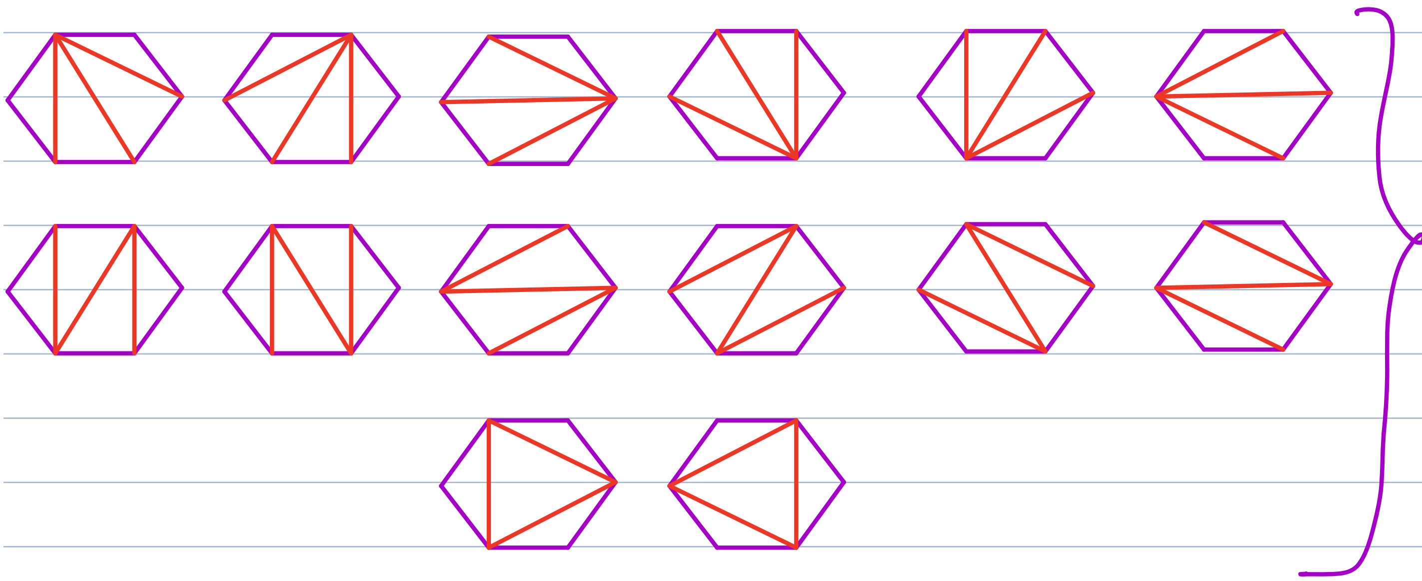


$G_1 = 1$

$G_2 = 2$

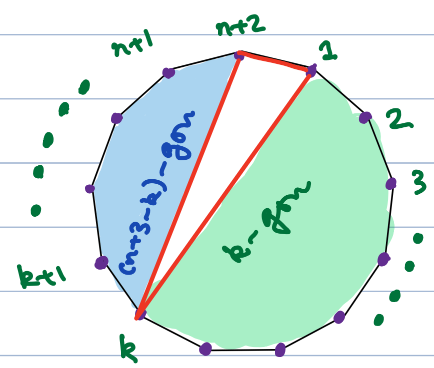


$G_3 = 5$



$G_4 = 14$

Let the vertices of our $(n+2)$-gon be labelled $1, 2, \ldots, n+1, n+2$. Fix $k \in \{1, 2, \ldots, n+1\}$. The vertices $1, n+2$, and k form a triangle (in the pic, the one with the red border). From the picture this gives us two other polygons; a k-gon and an $(n+3-k)$-gon.



We therefore have the relation

$$G_n = \sum_{k=2}^{n+1} G_{k-2}\, G_{n+1-k}$$

$$= \sum_{p=0}^{n-1} G_p\, G_{n-1-p} \qquad (p = k-2, \text{ i.e. } k = p+2)$$

In other words

$$\boxed{\begin{gathered} G_{n+1} = \sum_{p=0}^{n} G_p G_{n-p} \\ G_0 = 1, \ G_1 = 1 \end{gathered}}$$

This is the same recursive relation as the one that the Catalan numbers satisfy, and so

$$\boxed{G_n = C_n = \frac{1}{n+1} \binom{2n}{n}}$$