2.6 The Binomial Theorem

Theorem: Let $n \in \mathbb{N}$. Then
$$(x+y)^n = \sum_{n=0}^{n} \binom{n}{i} x^i y^{n-i}, \quad x, y \in \mathbb{R}.$$

Proof: Fix i. Write $(x+y)^n$ as $(x+y)(x+y)\cdots(x+y)$ (n factors)

$$(x+y)^n = \overbrace{(x+y)(x+y)\cdots(x+y)\cdots(x+y)}^{n\text{-times}}(x+y)$$

Colour i of the factors blue and the remaining $n-i$ red. Pick x's from the blue factors and y's from the red. In the expansion of the right side, this choice yields $x^i y^{n-i}$. There are $\binom{n}{i}$ such choices and so the # of $x^i y^{n-i}$ in the expansion is $\binom{n}{i}$. Since i varies from 0 to n, we see that
$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}.$$

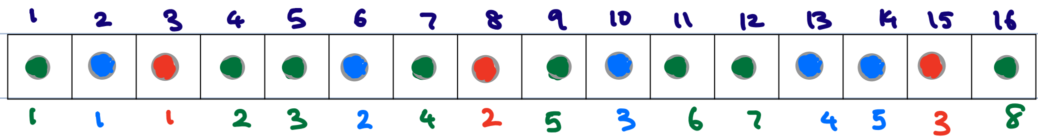
Corollary: $$2^n = \sum_{i=0}^{n} \binom{n}{i}$$

Proof: Set $x = y = 1$ in the binomial theorem above to get the answer.

The above is an algebraic proof. Here is a combinatorial proof. Consider the set of all binary strings of length n. There are 2^n of them. Fix $i \in \{0, 1, \ldots, n\}$. The number of binary strings of length n which have i 1's in them is $\binom{n}{i}$. Summing over i, we see that $\sum_{i=0}^{n} \binom{n}{i}$ is the total number of binary strings of length n. The result follows. //

2.7 Multinomial coefficients

Let X be a set with 16 elements. In how many ways can we pick three subsets, one of size 8, a second a second of size 3, and (of course) the last of size 5?



Here is how one could do it. Pick a subset of size 8. There are $\binom{16}{8}$ ways of doing this. Having picked such a subset, from the remaining elements, pick a subset of size 5. There are $\binom{8}{5}$ ways of doing this. Exactly 3 elements remaining, giving us the third subset.

The number of ways of doing this is clearly

$$\binom{16}{8}\binom{8}{5} = \frac{16!}{8!\,8!}\;\frac{8!}{5!\,3!} = \frac{16!}{8!\,5!\,3!}$$

Notations: Suppose $n \in \mathbb{N}$, $k_1, \dots, k_r \in \mathbb{N}_0$, with $k_1 + k_2 + \dots + k_r = n$. The number $\dfrac{n!}{k_1!\,k_2!\dots k_r!}$ is called a multinomial coefficient

and is denoted $\binom{n}{k_1, \dots, k_r}$.

Multinomial coefficient $\rightarrow \binom{n}{k_1, \dots, k_r} = \dfrac{n!}{k_1!\,k_2!\dots k_r!}$

Note that if $k_1 + k_2 = n$, then

$$\binom{n}{k_1, k_2} = \binom{n}{k}$$

This can cause confusion.

Theorem: Let $n \in \mathbb{N}$ and $k_1, k_2, \dots, k_r \in \mathbb{N}_0$ be such that $n = k_1 + k_2 + \dots + k_r$. The number of ways of splitting a set of size n into an ordered list of r disjoint subsets $A_1, A_2, \dots, A_r$ s.t. $|A_i| = k_i$ is

$$\frac{n!}{k_1!\,k_2!\dots k_r!} = \binom{n}{k_1, k_2, \dots, k_r}$$

Proof: The strategy is the same as before.

- From the set X, pick a subset A_1 of size k_1. There are $\binom{n}{k_1}$ ways of doing this.
- From the remaining $n-k_1$ elements, pick a subset of size k_2. There are $\binom{n-k_1}{k_2}$ ways of doing this.
- After picking A_1 and A_2, from the remaining $n-k_1-k_2$ elements pick a subset A_3 of size k_3. There are $\binom{n-k_1-k_2}{k_3}$ ways of doing.

Continue in this manner until you have subsets $A_1, A_2, \ldots, A_r$ with $|A_i| = k_i$, $i = 1, \ldots, r$. The number of ways of doing this is

$$\binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{n-k_1-k_2-\cdots-k_{r-1}}{k_r}$$

$$= \frac{n!}{k_1!\,\cancel{(n-k_1)}!}\;\frac{\cancel{(n-k_1)}!}{k_2!\,\cancel{(n-k_1-k_2)}!}\;\frac{\cancel{(n-k_1-k_2)}!}{k_3!\,\cancel{(n-k_1-k_2-k_3)}!}\cdots\frac{\cancel{(n-k_1-k_2-\cdots-k_{r-1})}!}{k_r!\,0!}$$

$$= \frac{n!}{k_1!\,k_2!\cdots k_r!} \quad //$$

Example: Let $0 \le k \le n-1$. Give a combinatorial proof of.

$$\binom{n}{k+1} = \sum_{i=k+1}^{n}\binom{i-1}{k} = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n-1}{k}$$

Proof: Let $i \in \{k+1, \ldots, n\}$. Pick a subset of size k from $[i-1] = \{1, 2, \ldots, i-1\}$. There are $\binom{i-1}{k}$ ways of doing this. Say $B = \{x_1, \ldots, x_k\}$ is picked from $\{1, \ldots, i-1\}$. Let $A = \{x_1, \ldots, x_k\} \cup \{i\}$. Then A is a subset of size $k+1$ in $[n]$. To go the other way, if A is a subset of size $k+1$ in $[n]$, then set i equal to the largest element in A. Clearly $k+1 \le i \le n$. Let $B = A - \{i\}$. Then B is a set of size k in $[i-1] = \{1, 2, \ldots, i-1\}$. It follows that for each $i \in \{k+1, \ldots, n\}$ we have $\binom{i-1}{k}$ subsets of $[n]$ of size $k+1$, with i the largest element in the set. Thus

$$\binom{n}{k+1} = \sum_{i=k+1}^{n} \binom{i-1}{k}.$$