Notations: $\mathbb{N} = \{1, 2, 3, \dots, n, \dots\}$ = set of natural numbers

$\mathbb{N}_0 = \{0, 1, 2, 3, \dots, n, \dots\} = \{0\} \cup \mathbb{N}$ = set of non-positive integers

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots, \pm n, \dots\}$ = set of integers

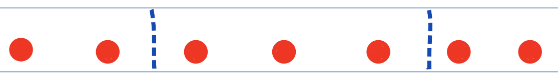
$\mathbb{R}$ = set of real numbers

$[n] = \{1, 2, 3, \dots, n\}$ for $n \in \mathbb{N}$.

2.5 The ubiquitous nature of binomial coefficients

Example: In how many ways can you distribute 7 apples to 3 people so that every person gets at least one apple? The apples are considered identical.

Soln:



} There are 6 gaps between the apples. Pick two and place bars there.

Lay out the seven apples in a line as above. There are six gaps between the apples. Pick two of them, e.g. the gaps identified by the two vertical bars. The two bars divide the apple into three groups. Give the left most group to the first person, the middle group to the second, and the last to the third. This gives us every possible way of distributing the seven apples.

There are 6 gaps and we picked 2. There are $\binom{6}{2}$ ways of doing this. Hence the number of ways of distributing the apples is

$$\binom{6}{2} = \frac{6 \cdot 5}{2} = 15 \text{ ways.}$$

Reformulation: The number of solutions to

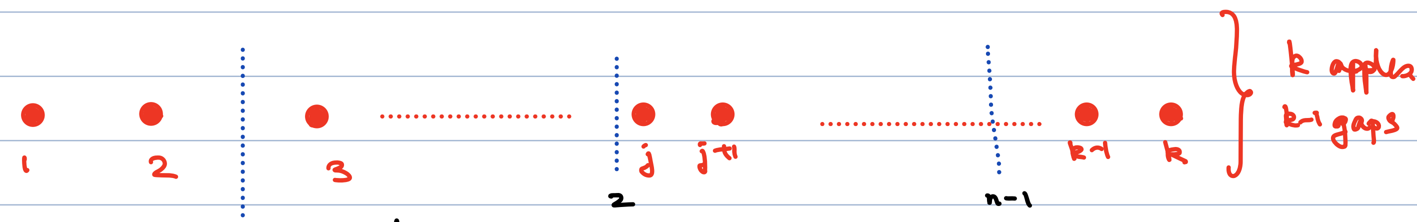
$$x_1 + x_2 + x_3 = 7, \quad x_i \in \mathbb{Z}, \quad x_i > 0$$

is 30.

General case: In how many ways can one distribute k identical apples to n people $(n \le k)$ so that every person gets at least one apple?

Solution: Use the same strategy as before. Lay out the k-apples in a line. Pick out $n-1$ gaps from the $k-1$ gaps. Any such choice groups the apples into n groups and vice-versa. As before, the answer is:

$$\binom{k-1}{n-1}.$$



pick $n-1$ bars.

Reformulation: The number of solutions to the equation

$$x_1 + x_2 + \dots + x_n = k$$

where each x_i is a positive integer (i.e. $x_i \in \mathbb{N}$ for $i = 1, \dots, n$) is $\binom{k-1}{n-1}$.

Example: How many ways are there of distributing k apples amongst n persons, with "no apples" being a valid allocation for a person.

Solution:

Suppose we have distributed the k apples to the n persons. Now give each person an extra apple. We have now distributed $n+k$ apples amongst n people so that everyone gets at least one apple. Conversely suppose we distribute $n+k$ apples amongst n people so that every person gets at least one apple. Now take away one apple from everyone. Then we have distributed k apples amongst n people without restrictions.

This gives:

# of ways of distributing k apples amongs n people
= # of ways of distributing $n+k$ apples amongs n person so that everyone get at least one apple
$= \binom{n+k-1}{n-1}$.

Reformulation. The number of solutions to
$$x_1 + x_2 + \dots + x_n = k$$
with each $x_i \in \mathbb{N}_0$, $i=1,\dots,n$ is $\binom{n+k-1}{n-1}$.

Example: How many solutions are there to
$$x_1 + x_2 + x_3 + x_4 + x_5 \le 300$$
with $x_i \in \mathbb{N}_0$? ($\mathbb{N}_0 = \{0, 1, 2, \dots, n, \dots\}$)

Solution:

This is the same as the number of solutions to
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 300$$

extra variables

with $x_i \in \mathbb{N}_0$, $i=1,\dots,5$.

In fact given a solution to the new problem, we have $x_1+x_2+x_3+x_4+x_5 = 300 - x_6 \le 300$ with $x_i \in \mathbb{N}_0$.
Conversely, given a solution to the old problem, then set $x_6 = 300 - (x_1+x_2+x_3+x_4+x_5)$. Then $x_6 \in \mathbb{N}_0$ and $x_1+x_2+x_3+x_4+x_5+x_6 = 300$, $x_i \in \mathbb{N}_0$, and we have a soln. to the new problem.

So the answer is $\binom{300+6-1}{6-1} = \binom{305}{5}$.

Example: How many ways can one distribute 7 apples to John, Paul, and Mary so that John and Mary get at least one apple, but Paul could get no apples?

Solution:

Distribute 8 apples among the three of them so that everyone gets at least one, and then take away one from Paul. There are $\binom{8-1}{3-1}$ ways of doing this.

Ans: $\binom{7}{2} = \frac{(7)(6)}{2!} = 21$

Reformulation: The number of solution to

$$x_1 + x_2 + x_3 = 7, \quad x_i \in \mathbb{Z}, \quad x_1, x_2 \geq 1, \quad x_3 \geq 0$$

is 21

Example: How many ways are there of buying 9 boxes of tea from a store which has 3 different varieties of tea, such that each box has only one variety of tea? (Assume the store has an inexhaustible supply of each variety of tea.)

Solution:

Suppose one picks b_1 boxes of the first variety, b_2 of the second, and b_3 of the third. Then we have

$$b_1 + b_2 + b_3 = 9, \quad b_i \in \mathbb{N}_0.$$

The number of solutions to this is $\binom{9+3-1}{3-1} = \binom{11}{2}$

Ans: $\binom{11}{2} = \frac{(11)(10)}{2} = 55.$

Theorem: The number of ways of choosing k objects from n objects with repetitions allowed is

$$\binom{n+k-1}{n-1}.$$

Proof: Suppose we pick x_1 of the first object, x_2 of the second, ..., x_n of the nth object. Then

$$x_1 + x_2 + \dots x_n = k \quad \text{with} \quad x_i \in \mathbb{N}_0.$$

Conversely, any solution to the above gives us a way of choosing k objects from n objects. Since the # of solns is $\binom{n+k-1}{n-1}$, the theorem is proved. //

Lattice Paths

Definition: A lattice path in the plane is a curve made up of line segments that either go from a point (i,j) to the point $(i+1, j)$ or from a point (i,j) to a point $(i, j+1)$.

Another definition, equivalent to the one above is that a lattice path in the plane is a sequence of pairs of integers

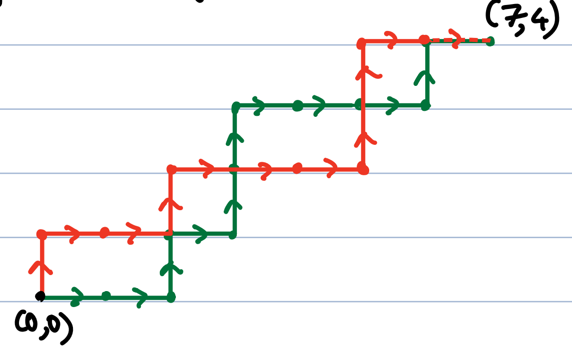
$$(m_0, n_0), (m_1, n_1), (m_2, n_2), \dots, (m_k, n_k)$$

such that for all $i = 1, \dots, k-1$, either

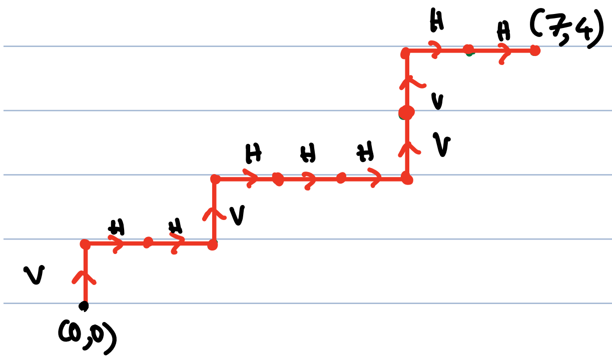
(a) $m_{i+1} = m_i + 1$ and $n_{i+1} = n_i$; or

(b) $m_{i+1} = m_i$ and $n_{i+1} = n_i + 1$.

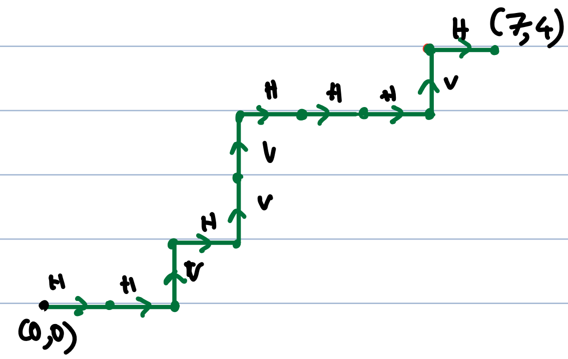
Let H be a unit horizontal move $((i,j) \mapsto (i+1, j))$ and V a unit vertical move. Let $X = \{H, V\}$. Then a lattice path is also an X-string together with an initial point (m_0, n_0). For example the red path below is the origin $(0,0)$ together with the string VHHVHHHVVHH and the green path is $(0,0)$ together with the string HHVHVVHHHV.



Question: Let $m, n \geq 0$. What is the number of lattice paths from $(0,0)$ to (m,n), $m, n \geq 0$.



11 moves. 7 H's, 4 V's.



11 moves. 7 H's, 4 V's.

Answer: A path from $(0,0)$ to (m,n) is the same as an X-string $(X = \{H, V\})$ of length $m+n$ with m H's in the string (or, equivalently, n V's). We have pick m places in a string of length $m+n$ to put the H's in. The answer is clearly

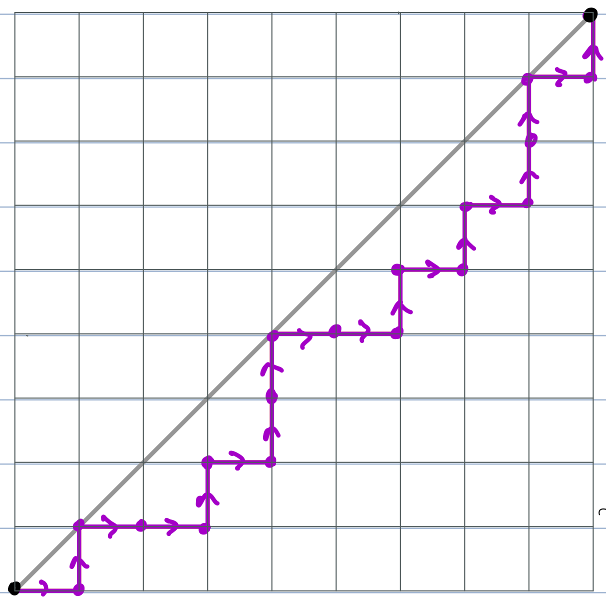
$$\binom{m+n}{m} = \binom{m+n}{n}.$$

Remark. Suppose $p, q, m, n \in \mathbb{Z}$ with $p \leq m$, $q \leq n$. Then the number of lattice paths from (p,q) to (m,n) is the same as the # of lattice paths from $(0,0)$ to $(m-p, n-q)$. Hence

$$\boxed{\#\text{ of lattice paths from } (p,q) \text{ to } (m,n) = \binom{m+n-p-q}{m-p} = \binom{m+n-p-q}{n-q}}$$

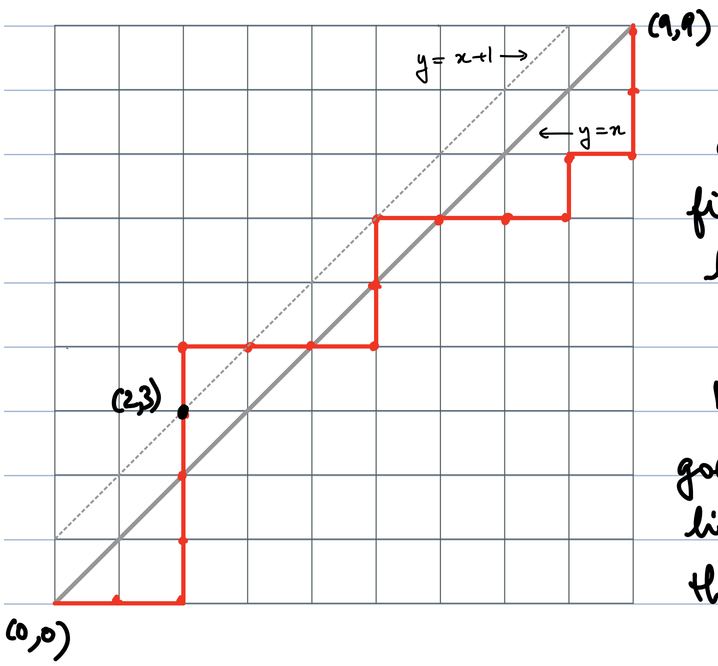
Example: How many lattice paths from $(0,0)$ to (n,n) are there which never go above the diagonal?

("The diagonal" is the line $y=x$).



The path displayed is one which never goes above the diagonal. It does touch the diagonal in many places though. It anyway has to at $(0,0)$ and (n,n).

H V HH V H V V HH V H V H V V H V ← (0,0)



A path which goes above the diagonal. The point $(2,3)$ is the first lattice point of the path which lies above the diagonal.

HH VVVV HHH VV HHH V H VV ← (0,0)

Note that every path to (n,n) which goes above the diagonal, must touch the line $y=x+1$. In the picture, $(2,3)$ is the first instance of this for our path.

Call a lattice path from $(0,0)$ to (n,n) good if it never goes above the diagonal. Otherwise, call it bad. The purple path above is good while the red one is bad. Let

$P =$ Set of all lattice paths from $(0,0)$ to (n,n)

$G =$ Set of good paths

$B =$ Set of bad paths.

Then

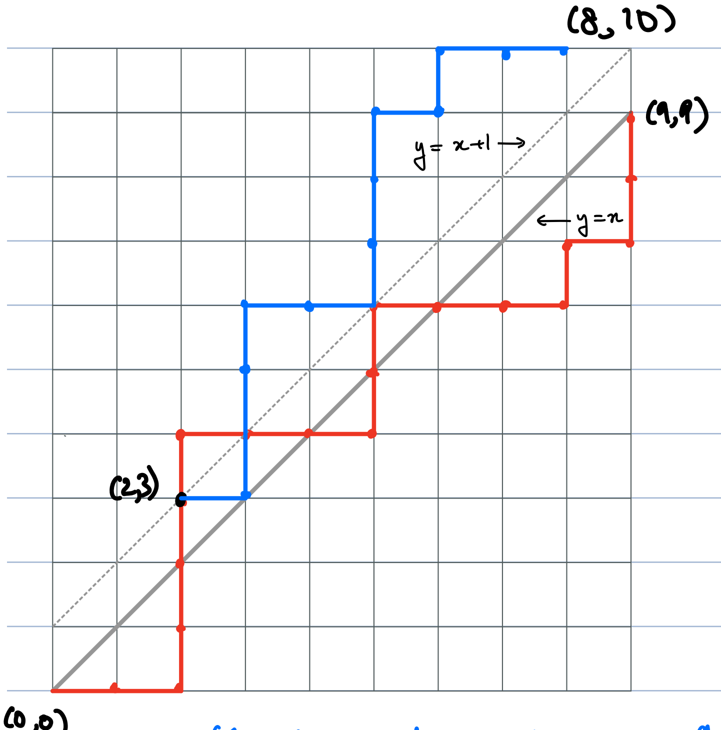
$$|P| = |G| + |B|$$

Since $|P| = \binom{2n}{n}$, this gives

$$|G| = |P| - |B| = \binom{2n}{n} - |B|.$$

Let us work out $|B|$.

Suppose σ is a bad path. Then there is an i, $0 \le i \le n$, such that $(i, i+1)$ is a point in the path. Let i be the smallest such number. Another way of saying this is that the bad path σ must hit the line $y = x+1$, and let $(i, i+1)$ be the first instance where it does. In the red path above, $(2,3)$ is the first instance where the path hits $y = x+1$ and $i = 2$. Suppose our bad path σ is as



$$\sigma = \sigma_1 \sigma_2$$

where σ_1 is the portion of σ from $(0,0)$ to $(i, i+1)$ and σ_2 is portion from $(i, i+1)$ to (n, n).

Let $\tilde{\sigma}_2$ be the path from $(i, i+1)$ to $(n-1, n+1)$ obtained by switching every horizontal move in σ_2 to a vertical move & every vertical move to a horizontal move.

Let

$$f(\sigma) = \sigma_1 \cdot \tilde{\sigma}_2 .$$

(In the picture above, $n = 9$ and $(i, i+1) = (2,3)$. The blue path is $\tilde{\sigma}_2$.) Note that in general, $\tilde{\sigma}_2$ is simply the reflection of σ_2 about the line $y = x+1$ and that $f(\sigma)$ is a path from $(0,0)$ to $(n-1, n+1)$. (Another way of seeing this is as follows: The path σ_2 has $n-i$ horizontal segments and $n-i-1$ vertical segments. This means that $\tilde{\sigma}_2$, its reflection about $y = x+1$, has $n-i$ vertical segments and $n-i-1$ horizontal segments. Since the initial point of $\tilde{\sigma}_2$ is $(i, i+1)$, its final point must have x-coordinate equal to $i + (n-i-1) = n-1$ and y-coordinate equal to $i+1+(n-i) = n+1$. Thus the terminal point of $\tilde{\sigma}_2$ and hence of $f(\sigma)$ is $(n-1, n+1)$.)

Conversely, given <u>any</u> path from τ from $(0,0)$ to

$(n-1, n+1)$, it must hit the line $y = x+1$. Let $(i, i+1)$ be the first first instance it does. Write $\tau = \tau_1 \tau_2$ where τ_1 is the part of τ from $(0,0)$ to $(i, i+1)$ and τ_2 the part from $(i, i+1)$ to $(n-1, n+1)$. Let $\tilde{\tau}_2$ be the reflection of τ_2 about $y = x+1$. Let

$$g(\tau) = \tau_1 \tilde{\tau}_2.$$

It is clear that $g(\tau)$ is a lattice path from $(0,0)$ to (n,n). (Apply the argument we gave to show that $f(\sigma)$ terminates at $(n-1, n+1)$.)

Moreover $g(\tau)$ is a bad path, since $(i, i+1)$ is a point on $g(\tau)$. (In the picture, if τ is the path from $(0,0)$ to $(8,10)$ which is the red bit from $(0,0)$ to $(2,3)$ followed by the blue path then τ_2 is the blue path and $\tilde{\tau}_2$ is the red bit from $(2,3)$ to $(9,9)$.) .)

Let Q be the set of lattice paths from $(0,0)$ to $(n-1, n+1)$. We have just shown that $f : B \longrightarrow Q$ is a bijective correspondence with inverse $g : Q \longrightarrow B$. Thus

$$|B| = |Q| = \binom{2n}{n-1} = \binom{2n}{n+1}.$$

It follows that

$$|G| = \binom{2n}{n} - \binom{2n}{n-1}$$

$$= \frac{(2n)!}{n!\, n!} - \frac{(2n)!}{(n-1)!\,(n+1)!}$$

$$= \frac{(2n)!}{n!\, n!} - \frac{n}{n+1} \frac{(2n)!}{n!\, n!}$$

$$= \left(1 - \frac{n}{n+1}\right) \frac{(2n)!}{n!\, n!}$$

$$= \frac{1}{n+1} \binom{2n}{n}.$$

The number $\frac{1}{n+1}\binom{2n}{n}$ is called a Catalan number and is denoted C_n.

$$\boxed{C_n = \frac{1}{n+1}\binom{2n}{n}}$$ ← Catalan number