Sep 13, 2022 | Lecture 2 | MAT344

Last lecture

- Strings.
- Permutations: $P(n, m) = \frac{n!}{(n-m)!}$ $(0 \le m \le n)$

  # of m-strings with distinct entries from a set of n elements.

  Special case $P(n,n) = n!$
- Examples:
  - Seating 4 people in a round table with 7 chairs $\longrightarrow \frac{6!}{3!}$
  - # of ways of arranging the letters of COMBINATORICS $\longrightarrow \frac{13!}{2!2!2!}$

2.3 Combinations (subsets)

$X = \{A, B, C, D, E, F, G, H\}$.

X- Permutations of length 4: ADEG, GDEA, FHCD, DHBF

X- Combinations of size 4: $\{A, D, E, G\}$, $\{C, D, F, H\}$, $\{D, H, F, B\}$

Definition: A combination of size k of a finite set X is a subset of X with k elements.

OVER $\longrightarrow$

Example: $X = \{A, B, C, D, E\}$, $k = 3$.

	A	B	C	D	E	
1	●	●	●			$\{A, B, C\}$
2	●	●		●		$\{A, B, D\}$
3	●	●		●		$\{A, B, E\}$
4	●		●	●		$\{A, C, D\}$
5	●		●		●	$\{A, C, E\}$
6	●			●	●	$\{A, D, E\}$
7		●	●	●		$\{B, C, D\}$
8		●	●		●	$\{B, C, E\}$
9		●		●	●	$\{B, D, E\}$
10			●	●	●	$\{C, D, E\}$

There are 10 of these.

Question: Is there a way of getting the answer without listing possibilities?

The answer is YES.

Trick: Figure out all permutations of length 3. Then account for the over counting. Given an X-string $s = x_1 x_2 x_3$, there is a subset of X

$$F(s) = \{x_1, x_2, x_3\}$$

associated with s. Suppose $s' = x_1' x_2' x_3'$ is another permutation of length 3. Then $F(s) = F(s')$ if and only if s' is a rearrangement of s. There are $3!$ rearrangements of s. So

$$P(5, 3) = 3!\,(\text{\# of combinations of size 3 from } X)$$

Notation: Let $C(n, k)$ denote the number of combinations of size k from a set of size n. $(0 \le k \le n)$

The above technique generalizes, and the same argument gives $P(n,k) = k!\, C(n,k)$.

Proposition 2.9: Let X be a finite set of size $n = |X| = \mathbb{N}$. The number of combinations of X of size $k \leq n$ is

$$C(n,k) = \frac{P(n,k)}{k!} = \frac{n!}{(n-k)!\,k!} =: \binom{n}{k} \longleftarrow \left\{ \begin{array}{l} \text{"n choose k"} \\ \text{"binomial coefficient"} \\ C(n,k). \end{array} \right.$$

2.4 Combinatorial Proofs.

Main Principle: Count in two different ways.

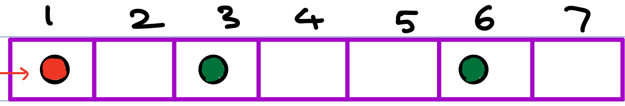
Examples:

1. Show
$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}, \quad 0 \leq k \leq n.$$

Soln: An algebraic proof is easy and left to you. But a combinatorial proof is more illuminating. The L.S. is the number of ways of choosing k elements from the set $[n] = \{1,2,3,\dots,n\}$. First count the number of subsets of size k containing 1, and then count the number of subsets not containing 1, and add the two results. Choosing k elements so that 1 is amongst the elements chosen is the same as choosing $k-1$ elements from $\{2,3,\dots,n\}$. There are $\binom{n-1}{k-1}$ ways of doing this.

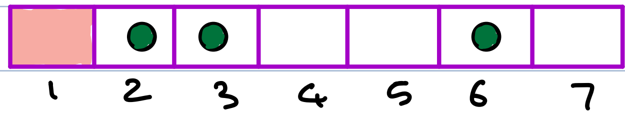
Choosing k elts so that 1 is NOT chosen is the same as picking k elts from $\{2,3,\dots,n\}$. There are $\binom{n-1}{k-1}$ ways of doing this.

1 is picked



Have to choose 2 elements from $\{2,3,4,5,6,7\}$

1 is NOT picked



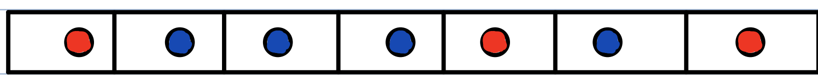
Have to choose 3 elements from $\{2,3,4,5,6,7\}$.

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

Those which do NOT contain 1

Those which do contain 1.

2. $$\binom{n}{k} = \binom{n}{n-k}$$



$n = 7$
$k = 3$

● = spot which is picked

● = spot which is not picked.

Solution:

Choosing k elements from a set X of size n to put in a bag is the same as choosing $n-k$ elements from X which will not be put in the bag.

3. $$\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \dots + \binom{n}{n}^2 = \binom{2n}{n}.$$

Note: Did NOT do this in class.

Solution:

Let $X = [2n] = \{\underbrace{1, 2, \dots, n}_{A}, \underbrace{n+1, \dots, 2n}_{B}\}$

Let $A = [n] = \{1, 2, \dots, n\}$ and $B = \{n+1, \dots, 2n\}$.

The right side is the number of subsets of size n in X. Picking a subset of size n from X is the same as picking some elements from A, say r of them $(0 \le r \le n)$, and the remaining $n-r$ from B.

$$\# \text{ of subsets of } A \text{ of size } r = \binom{n}{r}$$

$$\# \text{ of subsets of } B \text{ of size } n-r = \binom{n}{n-r}$$

$\Rightarrow$ # of subsets of X of size n with r elements in A

$$= \binom{n}{r}\binom{n}{n-r}$$

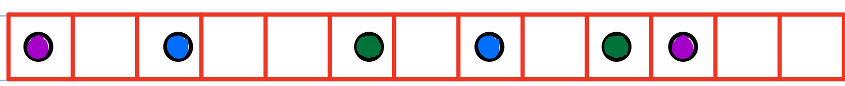
Now r varies from 0 to n. Hence

$$\# \text{ of subsets of } X \text{ of size } n = \sum_{r=0}^{n} \binom{n}{r}\binom{n}{n-r}$$

$$= \sum_{r=0}^{n} \binom{n}{r}^2 .$$

This is the left side.

Example: Recall the last example from Lecture I. The task was to find the number of "words" one can obtain by rearranging the letters in COMBINATORICS. Here is an idea for another proof. Once again, as in the last attempt at solving this, you are asked to flesh out the details. The idea is this: First pick 2 boxes out of the 13 in the set of boxes below (the blue spots)



Fill these with C's. Then pick, out of the remaining boxes, two boxes (green spots) and fill that with O's. Next pick two boxes (purple spots) out of the remaining boxes and fill them with I's. Distribute MBNATRS in the remaing boxes. Work out the answer yourself.