The closed form of a generating function

1. $1+x+x^2+\dots = \dfrac{1}{1-x}$

2. $1+x^2+x^4+\dots = \dfrac{1}{1-x^2}$

3. $1+\dfrac{x}{1!}+\dfrac{x^2}{2!}+\dots+\dfrac{x^n}{n!}+\dots = e^{nx}$

4. $\left(1+\dfrac{2x}{1!}+\dfrac{2^2x^2}{2!}+\dfrac{2^3x}{3!}+\dots\right)\left(1+3x+9x^2+27x^3+\dots\right) = \dfrac{e^{2x}}{1-3x}$

5. $\dfrac{x}{1!}+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\dots = \dfrac{e^x-e^{-x}}{2}$

In all these examples, the right side is a funtion which is obtained by taking finite sums, differences, products and quotients of well known funtions like:
(a) Polynomials, (b) e^{kx}, k a constant, ...
There are no infinite sums or products in the expressions on the right. These are called the closed forms of the functions on the left.

More examples

6. $1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\dfrac{x^6}{6!}+\dots = \dfrac{1}{2}\left(e^x+e^{-x}\right)$

7. $\displaystyle\sum_{n=0}^{\infty}\binom{n+k-1}{k-1}x^n = \frac{1}{(1-x)^k}$ $\qquad k\in\mathbb{N}_0$.

You will often be asked to write generating funtions or EGFs of sequences in closed form.

Probability Theory (Chapter 10)

We will only deal with finite sets or sometimes very special kinds of infinite sets (ones which are bijective to $\mathbb{N}$).

Before making formal definitions, some intuitive examples may be helpful.

Examples:

1. An urn contains four blue balls and five red balls. What is the probability that a ball chosen from the urn is blue? (Assume that each of the nine balls has an equal chance of being chosen.)

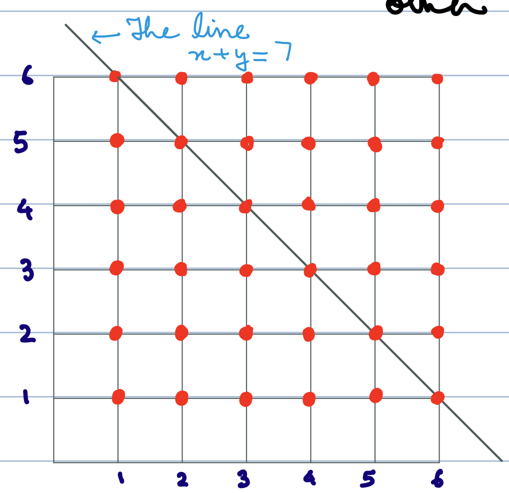
   Solution: Each of the nine outcomes is equally likely, and four of these produce a blue ball. So the answer is $4/9$.

2. What is the probability that when two dice are rolled, the sum of numbers on the two dice is 7?

When nothing is stated, one assumes that all outcomes are equally likely.

This is the "dangerous bend" symbol.

Solution: The outcomes can be identified with pairs (i,j), with $i,j \in \{1,2,\dots,6\}$. In other words there are 36 outcomes.



These are the 36 red dots in the xy-plane plotted in the picture on the left.

The "successful" outcomes are the pairs (i,j) amongst the red dots such that $i+j=7$. These are $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$. These precisely the red dots lying on the line $x+y=7$. There are six successful outcomes, and so the answer is $\frac{6}{36} = \frac{1}{6}$.

Probability spaces

A **probability space** is a pair (S, P) where S is a finite set and P is a function

$$P : \mathcal{P}(S) \longrightarrow [0,1]$$

where $\mathcal{P}(S)$ is the set of all subsets of S ($\mathcal{P}(S)$ = the "power set" of S) such that

1. $P(\phi) = 0$ and $P(S) = 1$
2. If A, B are subsets of S, and $A \cap B = \phi$, then $P(A \cup B) = P(A) + P(B)$.

2 Important: $P \neq \mathcal{P}$.

Suppose (S, P) is a probability space. Then:

(i) S is the **sample space**. (The book does not define this.)

(ii) P is the **probability measure**.

(iii) Subsets of S (i.e. elements of $\mathcal{P}(S)$) are called **events**. If $E \subset S$, then $P(E)$ is called the **probability of the event E**.

(iv) If x is an element of S then x is called an **outcome** (sometimes an "elementary outcome")

Examples:

3. In example 1 above, the sample space is
$$S = \{B_1, B_2, B_3, B_4, R_1, R_2, R_3, R_4, R_5\}$$
where B_i, $i = 1,2,3,4$ are the four blue balls and R_j, $j = 1,2,3,4,5$ are the five red balls. The event whose probability we have to compute is $E = \{B_1, B_2, B_3, B_4\}$. In other words, a successful outcome is that the ball x picked out of

the urn is in E. Since all outcomes are equally likely, and since

$$P(\{B_1\}) + P(\{B_2\}) + P(\{B_3\}) + P(\{B_4\}) + P(R_1) + P(R_2) + P(R_3) + P(R_4) + P(R_5)$$
$$= P(\{B_1, B_2, B_3, B_4, R_1, R_2, R_3, R_4, R_5\}) \quad \text{(by Property 2)}$$
$$= P(S)$$
$$= 1,$$

it follows that

$$P(\{x\}) = \frac{1}{9} \qquad \forall\ x \in S.$$

Thus $P(E) = P(\{B_1, B_2, B_3, B_4\}) \overset{\text{by Property 2}}{=} P(\{B_1\}) + P(\{B_2\}) + P(\{B_3\}) + P(\{B_4\})$
$$= \frac{4}{9}.$$

We don't have to be this elaborate in our solutions. It was done this way to illustrate the various definitions

4. In example 2 above $S = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid 1 \le i,j \le 6\}$, and
$E = \{(i,j) \in S \mid i+j = 7\} = \{(6,1), (5,2), (4,3), (3,4), (2,5), (1,6)\}$.
For $(i,j) \in S$,
$$P(\{(i,j)\}) = \frac{1}{36}$$

and hence
$$P(E) = \frac{6}{36} = \frac{1}{6}.$$

Remark: Since S is finite, P is completely specified by its values on outcomes. In other words, if one knows $P(\{x\})$ for every $x \in S$, then one knows $P(E)$ for every $E \in \mathcal{P}(S)$. To see this, suppose $E = \{x_1, \ldots, x_n\} \subset S$. Since $\{x_1\}, \{x_2\}, \{x_3\}, \ldots, \{x_n\}$ are mutually disjoint by Property 2 of probability spaces applied repeatedly, we see that
$$P(E) = P(\{x_1\}) + P(\{x_2\}) + \cdots + P(\{x_n\}).$$

Notational relaxation: If x is an outcome, it is simpler to write $P(x)$ rather than $P(\{x\})$ and we will often do so. Writing $P(\{x\})$ can be cumbersome.

Definition: An *experiment* is a procedure that yields one of a given set of possible outcomes.

This is a standard term in probability theory, but is not used in the textbook. It is however useful terminology. In example 1, the experiment is drawing a ball from the urn. In example 2, the experiment is rolling a pair of dice. The term allows a greater flexibility in describing a random process.

5. In a lottery, players win a large prize when they pick four digits that match, in the correct order, four digits selected by a random mechanical process. A smaller prize is won if only three digits are matched. What is the probability that player wins the large prize? What is the probability that a player wins the small prize?

Solution: The sample space is

$$S = \{0,1,\dots,9\} \times \{0,1,\dots,9\} \times \{0,1,\dots,9\} \times \{0,1,\dots,9\}.$$

Since nothing is stated about the probabilities of the outcomes, we will assume all outcomes are equally likely. There are 10^4 outcomes and so if $(x,y,z,t) \in S$ is an outcome then

$$P(\{(x,y,z,t)\}) = 10^{-4}.$$

There is only one way to choose all four digits correctly, and so the probability that a player wins the large prize is 10^{-4}.

To win the smaller prize one has to choose exactly three digits correctly, or, what is the same thing, one chooses exactly one digit incorrectly.

Let (a, b, c, d) be the outcome with all four digits correct. Let

$$E_1 = \{(x,y,z,t) \in S \mid x \neq a,\ y = b,\ z = c,\ t = d\}$$
$$E_2 = \{(x,y,z,t) \in S \mid x = a,\ y \neq b,\ z = c,\ t = d\}$$
$$E_3 = \{(x,y,z,t) \in S \mid x = a,\ y = b,\ z \neq c,\ t = d\}$$

and
$$E_4 = \{(x,y,z,t) \in S \mid x = a,\ y = b,\ z = c,\ t \neq d\}.$$

Then the choice (x,y,z,t) wins the smaller prize if and only if it lies in one of the sets E_1, E_2, E_3, E_4.

Thus one wins the smaller prize if and only if the event $E = E_1 \cup E_2 \cup E_3 \cup E_4$ occurs.

Now E_1, E_2, E_3, E_4 are mutually disjoint (i.e. $E_i \cap E_j = \phi$, if $i \neq j$). So

$$P(E) = P(E_1) + P(E_2) + P(E_3) + P(E_4).$$

Now for $i = 1, 2, 3, 4$

$$|E_i| = 9 \qquad \text{(why?)}$$

Hence

$$P(E_i) = \frac{9}{10^4}$$

which yields

$$P(E) = 4 \times \frac{9}{10^4} = \frac{36}{10^4} = \frac{9}{2500}.$$

← This is good enough. Don't simplify

If you are curious, $\frac{9}{2500} \approx 0.0036$. So the chances of winning the smaller prize are very low, and the chances of winning the larger prize even lower $(= 0.0001)$.

Bridge games: A standard deck of cards contains 52 cards consisting of four "suits", each suit having

13 cards each. The suits are named spades, hearts, clubs, and diamonds. The 13 cards in a suit have "values" or "denominations" which are

2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace
 ‖ ‖ ‖ ‖
 J Q K A

Note there are four cards of every value, e.g. there is a 7 in each of the four suits; in the jargon of card players, a 7 of spades, 7 of hearts, 7 of clubs, and 7 of diamonds.

A bridge game is played with two teams consisting of two players each. The cards of one of the players, called the "dummy" are known to the other three players. Each player has 13 cards (called the "hand" of the player) and the cards in each hand (except that of the dummy) are known only to the player who has that hand.

People in the same team (partners) sit opposite each other and every player has an opponent to their left and one to their right.

	N	
W		E
	S	

N and S are in the same team; E and W are in the same team.

6. In a bridge hand you and your partner (who is the dummy) have two aces between the two of you. What is the probability that one of your opponents has the remaining two aces?

<u>Solution:</u>

On the face of it, it seems as if the probability is $1/2$. We will see this not so (though the answer is close).

Let us first work out the chance that the opponent to your left has both the remaining aces. You and your partner account for 26 cards that you know. The opponent to the left has two aces (there is only one way this can happen) from the 26 cards with your opponents. This opponent has 11 more cards, and these have to be chosen from 24 cards (since the fate of two of the cards is known). So

\# of ways opponent on left has the remaining two aces

$$= \binom{24}{11}.$$

The # of possible hands opp. on left has $= \binom{26}{13}$

This implies:

$$\text{Probability opponent on left has remaining aces} = \frac{\binom{24}{11}}{\binom{26}{13}}$$

$$= \frac{24!}{11!\,\cancel{13!}} \; \frac{13!\,\cancel{13!}}{26!}$$

$$= \frac{24!\;13!}{11!\;26!}$$

$$= \frac{(13)(12)}{(26)(25)}$$

$$= \frac{1}{2}\;\frac{12}{25} = \frac{6}{25}$$

By symmetry, the probability that your opponent on the right has the two remaining aces is also $6/25$. Since the two events are mutually exclusive:

$$P(\text{One of the opps. has both remaining aces}) = \frac{12}{25}.$$

This can be verified by an independent computation of a probability of the "complementary event", namely the event that each of the opponents has one ace.

There are two ways to distribute the two remaining aces amongst the two opponents (obviously!). Of the remaining 24 cards, 12 have to be distributed to the opponent on the left (the remaining 12 go to the one on the right). So

$$\#\text{ of ways opps. have one remaining ace each} = 2\cdot\binom{24}{12}.$$

Hence

$$P(\text{Each opp. has exactly one remaining ace}) = \frac{2\cdot\binom{24}{12}}{\binom{26}{13}}$$

$$= 2\cdot\frac{24!}{12!\,12!}\quad\frac{13!\,13!}{26!}$$

$$= 2\,\frac{13^2}{(26)(25)}$$

$$= \frac{13}{25}$$

This means that

$$P(\text{One of the opps. has both remaining aces}) = 1 - \frac{13}{25} = \frac{12}{25}$$

exactly as before.

Remark: Note that both $\frac{12}{25}$ and $\frac{13}{25}$ are close to $\frac{1}{2}$. However the chance of each opponent having one ace each is slightly higher than the chance that one of them holds both the remaining aces.

If you wish to, you can put all this in a "probability space framework" as follows:

Let R be the set of cards which are not yours or your partner's (so $|R| = 26$). Let A_1, A_2 be the aces which are not with you or your partner. The sample space S is:

$$S = \{ H \mid H \subset R \text{ and } |H| = 13 \}.$$

S represents the set of possible hands of the opponent to your left.

The event that the opponent to your left holds both A_1 as well as A_2 as part of his or her hand is

$$E_L = \{ H \in S \mid A_1 \in H \text{ and } A_2 \in H \}.$$

The "event" that the opponent to your right has A_1 and A_2 is:

$$E_R = \{ H \in S \mid A_1 \notin H \text{ and } A_2 \notin H \}.$$

The event that one of your opponents holds both A_1 and A_2 is:

$$E = E_L \cup E_R.$$

Clearly $E_1 \cap E_2 = \phi$.

Since no information has been provided about the probability measure, one assumes that every outcome $H \in S$ is equally probable. Now

$$|S| = \binom{26}{13}.$$

It follows that

$$P(H) = \frac{1}{\binom{26}{13}} \quad \forall\, H \in S.$$

We will often <u>not</u> frame our solution in terms of probability spaces, especially when a direct attack is possible. Nevertheless, it is a useful framework which gives us conceptual clarity and we may fall back on it when the problem asked seems confusing.