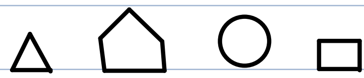
2.1 Strings (finite sequences)

Examples

- Strings of characters

$$\underbrace{t\,n\,u\,a\,b\,k\,l}_{\text{length } 7}, \quad \underbrace{v\,w\,x\,e\,g}_{\text{length } 5}$$

- String of polygons



length 4.

Definition: Let X be a set and $n \in \mathbb{N}$ ($\mathbb{N}$ = set of natural numbers). An X-string of length n is a function

$$s : \{1, 2, 3, \ldots, n\} \longrightarrow X.$$

Instead of writing s as above, we prefer to do the following. Let

$$x_i := s(i), \quad i = 1, \ldots, n. \qquad (\text{Note: } x_i \in X\,!)$$

Then we write

$$s = x_1 x_2 \cdots x_n. \qquad (\text{length is } n)$$

Remarks:

- Every "member" x_i of the string s is an element of X.
- Repetitions could occur (e.g. it is possible that x_1 equals x_{92}).
- Order matters! $\alpha\beta\gamma \neq \gamma\beta\alpha$.

Let $s = x_1 \ldots x_n$ be an X-string. The x_i are called the characters of s.

Definition (binary string): A binary string is a $\{0,1\}$-string.

(In other words $X = \{0,1\}$)

Examples: 1011011100, 00001010.

Question: How many binary strings of length n are there?

Answer: $\underbrace{2 \cdot 2 \cdots 2}_{n \text{ times}} = 2^n$

Theorem: The number of X-strings of length n is $|X|^n$, where $|X| = \#$ of elements in X.

Proof: This is best proved by induction which we will teach later. In the meanwhile here is a proof. For an arbitrary X-string $s = x_1 \dots x_n$ of length n, there are $|X|$ choices for x_1, $|X|$ choices for x_2, $|X|$ choices for x_3, and so on. So we have $|X| \cdots |X| = |X|^n$ choices in total. Distinct choices lead to distinct X-strings. So we have $|X|^n$ X-strings in total.

Example: A standard Ontario license plate is of the form

$$X_1 X_2 X_3 X_4 \cdot N_1 N_2 N_3$$

where X_1, X_2, X_3, X_4 are (upper case) letters of the English alphabet and N_1, N_2, N_3 are numbers between 0 and 9, i.e. $N_1, N_2, N_3 \in \{0,1,2,\dots,9\}$. How many such strings are there?

Answer: $26^4 \cdot 10^3$.

2.2 Permutations

Example: Let $X = \{a,b,c,d\}$.

Permutations: abc, adb, abcd, bd, dcab, bdac.

Not permutations: aab, aba, bbc, ddcb.

Definition: A permutation of a set X is an X-string of length n, $s = x_1 x_2 \dots x_n$, s.t. $x_i \neq x_j$ for $i \neq j$.
(This is the same as saying $s: \{1, 2, \dots, n\} \longrightarrow X$ is one-to-one).

Example: How many Ontario license plates are there with no repeated characters or numbers?

10 choices, 9 choices, 8 choices

$$X_1\ X_2\ X_3\ X_4 \cdot N_1\ N_2\ N_3$$

26 choices, 25 choices, 24 choices, 23 choices

Answer: $26 \cdot 25 \cdot 24 \cdot 23 \cdot 10 \cdot 9 \cdot 8 = \dfrac{26!}{22!} \cdot \dfrac{10!}{7!}$

$$= \frac{26!}{(26-4)!} \cdot \frac{10!}{(10-3)!}$$

Proposition 2.6 Let $m = |X|$. The number of permutations of X of length $n \leq |X|$ is

$$\boxed{P(m,n) = \frac{m!}{(m-n)!} = m(m-1)\cdots(m-(n-1))}$$

Proof: For a permutation $s = x_1 x_2 \dots x_n$ of length n, there are $m = |X|$ choices for x_1; having chosen x_1, there are $m-1$ choices for x_2; Once x_1 and x_2 are chosen, there are $m-2$ choices for x_3; and so on until there are $m-(n-1)$ choices for x_n. Hence there are $m(m-1)(m-2)\cdots(m-(n-1))$ choices. //

Example: Let $|X| = 16$. The number of permutations of length 4 is $16 \cdot 15 \cdot 14 \cdot 13$.

Important Special case: As above, let $|X| = m$. Then

$\#$ of permutations of X of length $m = P(m,m)$

$$= \frac{m!}{(m-m)!}$$

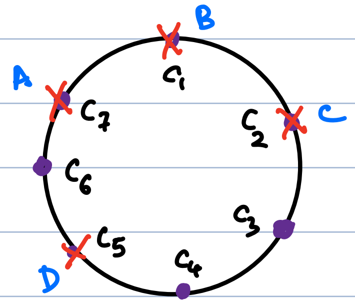
$$= \frac{m!}{0!}$$

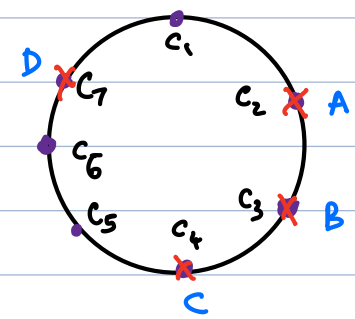
$$= \frac{m!}{1}$$

$$= m!$$

Example: In how many ways can you seat 4 different people at a round table with 7 chairs? Rotations are not counted as different.

E.g.





are considered the same. Here the set of chairs is $S = \{c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$ arranged clockwise, and the set of people is $P = \{A, B, C, D\}$

Solution: The number ways of assigning chairs to A, B, C, D so that two different persons are assigned distinct chairs is a one to one map

$$f : P \longrightarrow S.$$

In other words, this is simply a 4-permutation of the set S. There are $P(7,4) = \frac{7!}{3!}$ such maps. However, rotations are to be conted as the same. Each assignment is conted 7 times. Thus the solution is $P(7,4)/7 = 6!/3! = 6 \cdot 5 \cdot 4 = 120$.

Example: How many words of length 13 can you make with the letters in COMBINATORICS ?

Solution:

The difficulty is that there are two C's, two O's and two I's. The problem is solved by first overcounting and then understanding how much we have overcounted by, just as we did in the problem of counting seating arrangements in a round table.

So first pretend that the 13 letters shown are distinct. Then account for the repetitions (of C, O, and I). I will leave the working of the solution to you. If you are curious, the answer is $\frac{13}{2!2!2!}$. There is another way of doing this using the notion of "combinations". We will introduce that notion in the next class.