Generating functions and linear recurrence relations

Example (homogeneous case):

Consider the recurrence relation

$$a_n + a_{n-1} - 6a_{n-2} = 0, \quad n \geq 2; \quad a_0 = 1, \ a_1 = 3.$$

Re-write as

$$a_{n+2} + a_{n+1} - 6a_n = 0, \quad n \in \mathbb{N}_0; \quad a_0 = 1, \ a_1 = 3.$$

Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of $(a_n)_{n \geq 0}$.

We have

$$\sum_{n=0}^{\infty} (a_{n+2} + a_{n+1} - 6a_n) x^{n+2} = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} a_{n+2} x^{n+2} + x \sum_{n=0}^{\infty} a_{n+1} x^{n+1} - 6x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

i.e.

$$\sum_{n=2}^{\infty} a_n x^n + x \sum_{n=1}^{\infty} a_n x^n - 6x^2 \sum_{n=0}^{\infty} a_n x^n = 0$$

Note!

$$\Rightarrow \quad (g(x) - a_0 - a_1 x) + x(g(x) - a_0) - 6x^2 g(x) = 0$$

$$\Rightarrow \quad (1 + x - 6x^2) g(x) - a_0 - (a_1 + a_0)x = 0$$

Since $a_0 = 1$ and $a_1 = 3$, this gives

$$(1 + x - 6x^2) g(x) = 1 + 4x$$

i.e.

$$g(x) = \frac{1+4x}{1+x-6x^2}$$

$$= \frac{1+4x}{(1-2x)(1+3x)}$$

By the theory of partial fractions we get

$$\frac{1+4x}{(1-2x)(1+3x)} = \frac{A}{1-2x} + \frac{B}{1+3x}.$$

There are many ways to find A and B. The standard trick is to rewrite the above as

$$1+4x = A(1+3x) + B(1-2x),$$
and then equate coefficients of powers of x on both sides.
Another way is to set $x = 1/2$ and then $x = -1/3$
Setting $x = 1/2$ we get
$$1 + 4(1/2) = A\left(1 + \frac{3}{2}\right)$$
i.e. $$A = \frac{6}{5}$$

Setting $x = -1/3$ we get
$$1 - \frac{4}{3} = B\left(1 + \frac{2}{3}\right)$$

i.e. $$B = -1/5.$$
The other method (of equating coefficients) will also yield the same answer. In either case we get
$$g(x) = \frac{6}{5}\frac{1}{1-2x} - \frac{1}{5}\frac{1}{1+3x}$$
$$= \frac{6}{5}\sum_{n=0}^{\infty} 2^n \cdot x^n - \frac{1}{5}\sum_{n=0}^{\infty} (-3)^n x^n$$
$$= \sum_{n=0}^{\infty}\left\{\frac{6}{5}\cdot 2^n - \frac{1}{5}(-3)^n\right\} x^n.$$
It follows that
$$a_n = \frac{6}{5}2^n - \frac{1}{5}(-3)^n, \quad n \in \mathbb{N}_0.$$

Example (Non-homogeneous case):
Consider
$$a_n - a_{n-1} - 2a_{n-2} = 2^n, \quad n \geq 2$$
$$a_0 = 2, \; a_1 = 1.$$
Re-write this as
$$a_{n+2} - a_{n+1} - 2a_n = 2^{n+2}, \; n \in \mathbb{N}_0; \; a_0 = 2, \; a_1 = 1.$$

This gives

$$\sum_{n=0}^{\infty} \{a_{n+2} - a_{n+1} - 2a_n\} x^{n+2} = \sum_{n=0}^{\infty} 2^{n+2} \cdot x^{n+2}$$

i.e.

$$\sum_{n=0}^{\infty} a_{n+2} x^{n+2} - x \sum_{n=0} a_{n+1} x^{n+1} - 2x^2 \sum_{n=0} a_n x^n$$

$$= \sum_{n=0}^{\infty} 2^{n+2} \cdot x^{n+2}.$$

$$\Rightarrow \sum_{\underline{\underline{n=2}}}^{\infty} a_n x^n - x \sum_{\underline{\underline{n=1}}}^{\infty} a_n x^n - 2x^2 \sum_{\underline{\underline{n=0}}}^{\infty} a_n x^n = \sum_{n=2}^{\infty} 2^n x^n$$

Note

Let $g(x) = \sum_{n=0}^{\infty} a_n x^n$. Then the above translates to

$$\left(g(x) - a_0 - a_1 x\right) - x\left(g(x) - a_0\right) - 2x^2 g(x) = \sum_{n=0}^{\infty} 2^n x^n - 2x - 1$$

Since $a_0 = 2$ and $a_1 = 1$ the above gives

$$g(x) - 2 - x - xg(x) + 2x - 2x^2 g(x) = \frac{1}{1-2x} - 2x - 1$$

$$\Rightarrow (1 - x - 2x^2) g(x) - 2 + x = \frac{1}{1-2x} - 2x - 1$$

$$\Rightarrow \quad (1-2x)(1+x) g(x) = \frac{1}{1-2x} - 3x + 1$$

$$= \frac{2 - 5x + 6x^2}{1-2x}$$

Thus

$$g(x) = \frac{2 - 5x + 6x^2}{(1-2x)^2 (1+x)}$$

We have a partial fraction decomposition

$$\frac{2-5x+6x^2}{(1-2x)^2(1+x)} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1+x}$$

i.e.

$$2-5x+6x^2 = A(1-2x)(1+x) + B(1+x) + C(1-2x)^2 \quad \text{---} (*)$$

A, B, and C can be found by equating coefficients.
Another method is:

Set $x = 1/2$ in $(*)$. We get

$$2-5\left(\tfrac{1}{2}\right)+6\left(\tfrac{1}{4}\right) = B\left(1+\tfrac{1}{2}\right)$$

which gives

$$B = \frac{2}{3}.$$

Next set $x=-1$ in $(*)$. Get

$$2-5(-1)+6(-1)^2 = C(1-2(-1))^2$$

This gives

$$C = 13/9.$$

Next differentiate both sides of $(*)$ to get

$$-5+12x = A(1-2x) - 2A(1+x) + B - 4C(1-2x)$$

Now set $x = 1/2$ in the above. We get

$$1 = -2A\left(\tfrac{3}{2}\right) + \frac{2}{3}$$

$$\Rightarrow \quad A = -1/9$$

This yields

$$g(x) = \left(-\frac{1}{9}\right)\frac{1}{1-2x} + \frac{2}{3}\frac{1}{(1-2x)^2} + \frac{13}{9}\frac{1}{1+x}$$

$$= -\frac{1}{9}\sum_{n=0}^{\infty} 2^n x^n + \frac{2}{3}\sum_{n=0}^{\infty}(n+1)2^n x^n + \frac{13}{9}\sum_{n=0}^{\infty}(-1)^n x^n$$

$$= \sum_{n=0}^{\infty}\left\{\left(-\frac{1}{9}+\frac{2}{3}\right)2^n + \frac{2}{3}n\cdot 2^n + \frac{13}{9}(-1)^n\right\}x^n$$

$$= \sum_{n=0}^{\infty}\left\{\frac{5}{9}\cdot 2^n + \frac{2}{3}n\cdot 2^n + \frac{13}{9}(-1)^n\right\}x^n$$

Thus

$$a_n = \underbrace{\frac{5}{9}2^n + \frac{13}{9}(-1)^n}_{\text{soln of assoc. homog. eqn.}} + \frac{2}{3}n\cdot 2^n$$