Non-homogeneous linear recurrence relations

Example: Consider the recurrence relation

$$a_{n+2} - 5a_{n+1} + 6a_n = n, \qquad n \in \mathbb{N}_0.$$

If $\Phi(x) = x^2 - 5x + 6$, then the above eqn is the same as the equation

$$\Phi(A) f = F$$

where

$$F(n) = n, \qquad n \in \mathbb{N}_0.$$

There are many solutions to this equation, but let us find one. An intelligent guess for a solution is g_p, where

$$g_p(n) = \alpha n + \beta, \qquad n \in \mathbb{N}_0,$$

α, β being constant to be determined (so that $\Phi(A) g_p = F$).

The subscript "p" in g_p will be explained shortly.

We want $\Phi(A) g_p$ to equal F, i.e. we want.

$$(A^2 - 5A + 6) g_p = F$$

Now

$$A^2 g_p(n) = g_p(n+2)$$

$$= \alpha(n+2) + \beta$$

$$= \alpha n + 2\alpha + \beta$$

$$-5A g_p(n) = -5 g_p(n+1)$$

$$= -5\alpha(n+1) - 5\beta$$

$$= -5\alpha n - 5\alpha - 5\beta$$

$$6 g_p(n) = 6\alpha n + 6\beta$$

Thus

$$\Phi(A) g_p(n) = (\alpha - 5\alpha + 6\alpha) n + (2\alpha + \beta - 5\alpha - 5\beta + 6\beta).$$

$$= 2\alpha n - 3\alpha + 2\beta$$

This means $\Phi(A) g_p = F$ if and only if

$$2\alpha n - 3\alpha + 2\beta = n, \qquad n \in \mathbb{N}_0.$$

In other words (by equating coefficients)

$$2\alpha = 1 \quad \text{and} \quad -3\alpha + 2\beta = 0$$

This yields

$$\alpha = \frac{1}{2}, \quad \beta = \frac{3}{4}.$$

So $\quad g_p(n) = \frac{1}{2} n + \frac{3}{4}, \quad n \in \mathbb{N}_0.$

Thus we have found one solution, a particular solution. This explains the subscript "p" in g_p.

The homogeneous equation

$$\Phi(A) f = 0$$

is called the associated homogeneous equation.

Suppose g_1 is any solution of the assoc. homog. eqn. Then $\Phi(A) g_1 = 0$. Hence if

$$g = g_1 + g_p$$

then

$$\begin{aligned} \Phi(A) g &= \Phi(A)(g_1 + g_p) \\ &= \Phi(A) g_1 + \Phi(A) g_p \\ &= 0 + F \\ &= F. \end{aligned}$$

It follows that $g = g_1 + g_p$ is a solution of the original recurrence equation $\Phi(A) f = F$.

Conversely, suppose g is a soln. of $\Phi(A) f = F$. Set $g_1 = g - g_p$, so that $g = g_1 + g_p$.

Claim: g_1 is a solution of the assoc. homog. eqn.

Pf. of claim: $\Phi(A) g_1 = \Phi(A)(g - g_p) = \Phi(A) g - \Phi(A) g_p = F - F = 0.$

q.e.d. for claim

We know that $\Phi(x) = x^2 - 5x + 6$ can be factored as $\Phi(x) = (x-3)(x-2)$ and hence the general solution of the associated homogeneous equation $\Phi(A)f = 0$ is

$$g_h(n) = c\,2^n + d\,3^n, \qquad n \in \mathbb{N}_0,$$

where c, d are arbitrary complex constants.

From our discussion above the general solution to

$$a_{n+2} - 5a_{n+1} + 6a_n = n, \qquad n \in \mathbb{N}_0$$

is $g = g_h + g_p$, i.e.,

$$g(n) = c\cdot 2^n + d\cdot 3^n + \frac{1}{2}n + \frac{3}{4}$$

where $c, d \in \mathbb{C}$ are arbitrary constants.

* ——— * ——— * (End of example)

The argument given in the example for showing that every solution is of the form $g_h + g_p$, where g_h is the general solution of the homogeneous part and g_p is a particular solution is not specific to the example. We expand on this below.

General and particular solutions:

Let $\Phi(x)$ be a polynomial, $F: \mathbb{N}_0 \longrightarrow \mathbb{C}$ a function. Consider the recurrence relation

$$\Phi(A)f = F. \qquad \text{——— } (*)$$

The equation

$$\Phi(A)f = 0 \qquad \text{——— } (**)$$

is called the associated homogeneous equation. The argument given in the example works without change to give that if g_p is a specific solution of the eqn. $(*)$ then all other solutions are of the form

$$g = g_h + g_p$$

where g_h is a solution of (**), i.e. of the associated homogeneous equation. Thus if g_h is the general solution of (**) then the general soln. of (*) is $g_h + g_p$.

More examples

In examples 2, 3, and 4 below, we assume $\Phi(x) = x^2 - 5x + 6$. We will solve

$$\Phi(A) f = F$$

for specific $F : \mathbb{N}_0 \to \mathbb{C}$ in these example. In the example above, $F(n) = n$, $n \in \mathbb{N}_0$.

2. $F(n) = 7^n$, $n \in \mathbb{N}_0$.

Try $g_p(n) = \alpha \cdot 7^n$, $n \in \mathbb{N}_0$

$$\left.\begin{aligned} A^2 g_p(n) &= g_p(n+2) = \alpha \cdot 7^{n+2} = 49\alpha \cdot 7^n \\ -5A g_p(n) &= -5 g_p(n+1) = -5\alpha \cdot 7^{n+1} = -35\alpha \cdot 7^n \\ 6 g_p(n) &= 6\alpha \cdot 7^n . \end{aligned}\right\} \quad n \in \mathbb{N}_0$$

Thus

$$\begin{aligned} \Phi(A) g_p(n) &= (49\alpha - 35\alpha + 6\alpha) \cdot 7^n \\ &= 20\alpha \cdot 7^n \end{aligned}$$

for $n \in \mathbb{N}_0$.

Since $F(n) = 7^n$, $20\alpha \cdot 7^n = 7^n$, $\forall n \in \mathbb{N}_0$.

$$\Leftrightarrow \alpha = \frac{1}{20}$$

Thus $g_p(n) = \frac{1}{20} 7^n$

The general soln. is

$$g(n) = \underbrace{c\, 2^n + d\, 3^n}_{\substack{\text{homogeneous part} \\ g_h .}} + \frac{1}{20} 7^n , \quad n \in \mathbb{N}_0$$

with c, d being arbitrary complex constants.

3. $F(n) = n\,7^n, \quad n \in \mathbb{N}_0$

poly. of deg 1.

Try

$$g_p(n) = \underbrace{(\alpha n + \beta)}_{\text{poly. of deg 1.}} \cdot 7^n, \quad n \in \mathbb{N}_0$$

$$A^2 g_p(n) = g_p(n+2) = \{\alpha(n+2) + \beta\} \cdot 7^{n+2}$$
$$= 49\,(\alpha n + 2\alpha + \beta) \cdot 7^n$$
$$-5A\, g_p(n) = -5 g_p(n+1) = -5\{\alpha(n+1) + \beta\} \cdot 7^{n+1}$$
$$= -35\,(\alpha n + \alpha + \beta) \cdot 7^n$$
$$6 g_p(n) = (6\alpha n + 6\beta) \cdot 7^n.$$

Thus

$$\Phi(A)\, g_p(n) = \{(49\alpha - 35\alpha + 6\alpha) n + 98\alpha + 49\beta - 35\alpha - 35\beta + 6\beta\} \cdot 7^n$$

$$= \{20\alpha n + 63\alpha + 20\beta\} \cdot 7^n$$

$\Rightarrow \quad 20\alpha = 1 \quad$ and $\quad 63\alpha + 20\beta = 0$

$\Rightarrow \quad \alpha = \frac{1}{20} \quad$ and $\quad \beta = \frac{-63}{400}.$

$$g_p(n) = \left(\frac{1}{20} n - \frac{63}{100}\right) 70^n, \quad n \in \mathbb{N}_0$$

and gen'l soln is

$$g(n) = c\,2^n + d\,3^n + \left(\frac{1}{20} n - \frac{63}{100}\right) 70^n, \quad n \in \mathbb{N}_0$$

where c, d, arbitrary complex constants.

4. $F(n) = n^2 \cdot 2^n, \quad n \in \mathbb{N}_0.$

The first thought might be to try $g_p(n) = (\alpha n^2 + \beta n + \gamma) \cdot 2^n$, $n \in \mathbb{N}_0$. This will not work because $\gamma \cdot 2^n$ is already a solution of the associated homogeneous equation.

Instead one tries

$$g_p(n) = n \cdot (\alpha n^2 + \beta n + \gamma) \cdot 2^n, \quad n \in \mathbb{N}_0.$$

This will work. Details are left to you.

5. In this example we take

$$\Phi(x) = (x-2)^2 = x^2 - 4x + 4$$

and give the form of g_p for the eqn

$$\Phi(A) f = F$$

where

$$F(n) = n^3 \cdot 2^n \qquad n \in \mathbb{N}_0.$$

The n^3 factor asks us to look for g_p of the form $g_p(n) = (\alpha n^3 + \beta n^2 + \gamma n + \delta) \cdot 2^n$, but this will not work. We

have to account for the fact that 2 is a root of the characteristic polynomial of multiplicity 2. The correct form of g_p is

$$g_p(n) = n^{2} (\alpha n^3 + \beta n^2 + \gamma n + \delta) 2^n, \quad n \in \mathbb{N}_0$$

multiplicity of the root of the characteristic poly.

One has to solve for α, β, γ, and δ using the equation(s)

$$g_p(n+2) - 5 g_p(n+1) + 6 g_p(n) = n^3 \cdot 2^n, \quad n \in \mathbb{N}_0$$

The details are left to you.

The following theorem gives the general technique for finding g_p for equations of the form

$$\Phi(A) f = F$$

where $\Phi(x)$ is a polynomial and F is of the form

$$F(n) = (c_0 n^d + c_1 n^{d-1} + \dots + c_{d-1} n + c_d) \cdot \lambda^n, \quad n \in \mathbb{N}$$

where $c_0, c_1, \dots, c_d, \lambda$ are complex constants. If λ is a root of $\Phi(x)$ then its multiplicity (as a root) plays a role in the form of g_p that we choose. Here is the complete statement.

Theorem: Let $\Phi(x)$ be a polynomial and let $F: \mathbb{N}_0 \longrightarrow \mathbb{C}$ be the function

$$F(n) = (c_0 n^d + c_1 n^{d-1} + \dots + c_{d-1} n + c_d) \cdot \lambda^n, \quad n \in \mathbb{N}_0$$

where λ, c_i, $i = 0, \dots, d$ are complex constants with c_0 and λ non-zero.

Consider the recurrence relation

$$\Phi(A) f = F \quad \text{———} \quad (*).$$

(a) If λ is NOT a root of the characteristic polynomial Φ, then we can find a particular solution g_p of the form

$$g_p(n) = (\alpha_0 n^d + \alpha_1 n^{d-1} + \dots + \alpha_{d-1} n + \alpha_d) \cdot \lambda^n, \quad n \in \mathbb{N}_0$$

where $\alpha_0, \alpha_1, \dots, \alpha_d \in \mathbb{C}$.

(b) If λ IS a root of the characteristic polynomial Φ, say of multiplicity m, then we can find a particular solution g_p of the form

$$g_p(n) = n^m (\alpha_0 n^d + \alpha_1 n^{d-1} + \dots + \alpha_{d-1} n + \alpha_d) \cdot \lambda^n, \quad n \in \mathbb{N}_0.$$

Remarks:

1. The constants $\alpha_0, \alpha_1, \dots, \alpha_d$ are worked out from the relation

$$(\Phi(A) g_p)(n) = (c_0 n^d + c_1 n^{d-1} + \dots + c_{d-1} n + c_d) \cdot \lambda^n, \quad n \in \mathbb{N}.$$

2. For those who have seen some abstract algebra, the $g = g_h + g_p$ relationship is a special case of the fact that if $\phi: G \longrightarrow H$ is a group homomorphism, K the kernel of ϕ, then for $h \in \phi(G) \subset H$, $\phi^{-1}(h)$ is the coset Kg of K, where g is any element which maps to h under ϕ. In our case, $G = H = V$, $\phi = \phi(A)$, $h = F$, $g = g_p$, $K = S$, the space of solutions of $\Phi(A) f = 0$.

Ignore this comment if you haven't seen abstract algebra.

Recurrence relations and generating functions

Example: Consider the recurrence relation
$$a_{n+1} = \sum_{k=0}^{n} a_k a_{n-k} \quad , \qquad n \in \mathbb{N}_0$$
with initial condition
$$a_0 = 1.$$

Recall that the number C_n of Catalan paths satisfies this recurrence (as well as the initial condition). See Lecture 5, especially the section on Catalan paths. In Lecture 3, using the clever idea of reflections, we showed that the number of Catalan paths is $\frac{1}{n+1}\binom{2n}{n}$ (look up the tail end of Lecture 3).

We now solve this using generating functions. Let
$$g(x) = \sum_{n=0}^{\infty} a_n x^n$$
be the generating function for $(a_n)_{n=0}^{\infty}$.

We have
$$a_{n+1} = \sum_{k=0}^{n} a_k a_{n-k} \quad , \qquad n \in \mathbb{N}_0.$$

Multiplying both sides with x^{n+1} and summing we get
$$\sum_{n=0}^{\infty} a_{n+1} x^{n+1} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^{n+1}$$
i.e.
$$g(x) - a_0 = x \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k a_{n-k} \right) x^n$$
$$= x\, g(x)^2,$$
by the formula for the product of power series. Since $a_0 = 1$, the above can be re-written as

$$x\, g(x)^2 - g(x) + 1 = 0 .$$

This gives

$$g(x) = \frac{1 \pm \sqrt{1-4x}}{2x} .$$

Using L'hôpital's rule, we see that

$$\lim_{x\to 0} \frac{1+\sqrt{1-4x}}{2x} = -1 \quad \text{and} \quad \lim_{x \to 0} \frac{1-\sqrt{1-4x}}{2x} = 1 .$$

(Check these!)

Since $g(0) = a_0 = 1$, it follows that

$$g(x) = \frac{1 - \sqrt{1-4x}}{2x} .$$

Now, from Newton's binomial theorem we have

$$(1-4x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n$$

$$= 1 + \sum_{n=1}^{\infty} \binom{1/2}{n} (-1)^n 4^n x^n .$$

Moreover, we showed in Lecture 16 (see Corollary on page 3 of the notes for that lecture) that

$$\binom{1/2}{n} = \frac{1}{2n} \frac{(-1)^{n-1}}{4^{n-1}} \binom{2(n-1)}{n-1} , \quad n \geq 1$$

Thus

$$(1+4x)^{1/2} = 1 + \sum_{n=1}^{\infty} \frac{1}{2n} \frac{(-1)^{n-1}}{4^{n-1}} \binom{2(n-1)}{n-1} (-1)^n 4^n x^n$$

$$= 1 - \sum_{n=1}^{\infty} \frac{2}{n} \binom{2(n-1)}{n-1} x^n$$

$$= 1 - 2 \sum_{m=0}^{\infty} \frac{2}{m+1} \binom{2m}{m} x^{m+1} \qquad \left(\begin{array}{c} \text{Substitution:} \\ m = n-1 \end{array} \right)$$

Thus

$$\sqrt{1-4x} = 1 - 2x \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n .$$

Substitute this in the formula $g(x) = (1-\sqrt{1-4x})/2x$. Get

$$g(x) = \sum_{n=0}^{\infty} \frac{1}{n+1} \binom{2n}{n} x^n .$$

Thus

$$a_n = \frac{1}{n+1} \binom{2n}{n}, \qquad n \in \mathbb{N}_0$$

exactly as before!