Recurrence (Chapter 9)

Example: Let $g: \mathbb{N}_0 \longrightarrow \mathbb{C}$ be a function, and c a non-zero complex number. Consider the recurrence relation

$$a_{n+1} - c\, a_n = g(n) \quad , \; n \in \mathbb{N}_0$$

It is clear that if one knows a_0, then a_n for $n \geq 1$ can be determined. For example, $a_1 = ca_0 + g(0)$, $a_2 = ca_1 + g(1)$ etc.

Suppose $g(n) = 0$ $\forall\, n \in \mathbb{N}_0$. Then it is easy to see that $a_n = a_0 c^n$, $n \in \mathbb{N}_0$. Indeed if $n = 0$, this is a tautology. Suppose it is true for some $n \in \mathbb{N}_0$. Then

$$a_{n+1} = c\, a_n = c\,(a_0\, c^n) = a_0\, c^{n+1}.$$

Equations of the form $a_{n+1} - c a_n = 0$, $n \in \mathbb{N}_0$ are called linear homogeneous equations.

The advancement operator: A sequence of complex numbers $(a_n)_{n \geq 0}$ can be regarded as a function $f: \mathbb{N}_0 \longrightarrow \mathbb{C}$, namely the function $f(n) = a_n$, for $n \in \mathbb{N}_0$. We can define the advancement operator $A: V \longrightarrow V$ where V is the set of functions of the form $f: \mathbb{N}_0 \longrightarrow \mathbb{C}$, by the rule $Af: \mathbb{N}_0 \longrightarrow \mathbb{C}$ is the function

$$(Af)(n) = f(n+1).$$

In this notation $A^k f$ is the function $(A^k f)(n) = f(n+k)$, for all $k \in \mathbb{N}_0$. In particular A^0 is the identity on V.

In these terms the relation $a_{n+1} - c a_n = 0$ becomes $(A - c) f = 0$, and the solution is $f(n) = d\, c^n$, $n \in \mathbb{N}_0$, where $d \in \mathbb{C}$ is a constant $(d = a_0)$.

The following example is at the next level of complication.

Example: Consider the recurrence relation

$$a_{n+2} - 5a_{n+1} + 6a_n = 0 \quad , \quad n \in \mathbb{N}_0.$$

If we know a_n for two successive n's, says a_p and

a_{p+1}, then we know the next a_n, namely a_{p+2}, since $a_{p+2} = 5a_{p+1} - 6a_p$. It follows that one knows a_{p+k} for $k \geq 0$, by repeatedly using the recurrence relation above. In particular, if we know a_0 and a_1, then we know all the a_n, $n \in \mathbb{N}_0$.

As before, identifying $(a_n)_{n\geq 0}$ with $f: \mathbb{N}_0 \longrightarrow \mathbb{C}$ given by $f(n) = a_n$, the above relation can be rewritten as

$$(A^2 - 5A + 6)f = 0.$$

i.e., as

$$(A-3)(A-2)f = 0.$$

Note that if g_1 and g_2 are solutions of $(A^2 - 5A + 6)f = 0$ then for $a, b \in \mathbb{C}$, $ag_1 + bg_2$ is also a solution, for

$$(A^2 - 5A + 6)(ag_1 + bg_2) = a(A^2 - 5A + 6)g_1 + b(A^2 - 5A + 6)g_2 = 0.$$

Suppose g is a solution of

$$(A-2)f = 0$$

and h a solution of

$$(A-3)f = 0.$$

Then

$$(A^2 - 5A + 6)g = (A-3)(A-2)g = (A-3)(0) = 0$$

and

$$(A^2 - 5A + 6)h = (A-2)(A-3)h = (A-2)(0) = 0.$$

This means g and h are solutions of the original recurrence equation $a_{n+2} - 5a_{n+1} + 6a_n = 0$. Further, we know that

$$g(n) = p\,2^n, \quad n \in \mathbb{N}$$

and

$$h(n) = q\,3^n, \quad n \in \mathbb{N}$$

for some complex numbers p and q. Thus

$$a_n = p \cdot 2^n + q \cdot 3^n, \quad n \in \mathbb{N}_0$$

is a solution of our recurrence relation. From our above observations, this is a solution for every value of p and q in $\mathbb{C}$.

Now suppose that we also had initial conditions

$$a_0 = \alpha, \quad a_1 = \beta.$$

Then, as we just argued, these initial conditions completely determine the solution to the recurrence relation. Note that if $p\,2^n + q\,3^n$ is our solution, then

$$p + q = \alpha \quad \text{and} \quad 2p + 3q = \beta.$$

This means

$$p = 3\alpha - \beta \quad \text{and} \quad q = \beta - 2\alpha.$$

Thus $(3\alpha - \beta)\cdot 2^n + (\beta - 2\alpha)\cdot 3^n$ is the unique solution to our recurrence relation with initial conditions $a_0 = \alpha$, $a_1 = \beta$.

In particular, every solution of the recurrence relation

$$a_{n+2} - 5a_{n+1} + 6 = 0$$

is of the form

$$a_n = p\cdot 2^n + q\cdot 3^n, \quad n \in \mathbb{N}_0.$$

Example: Now consider the example

$$a_{n+2} - 2r\,a_{n+1} + r^2 a_n = 0, \quad n \in \mathbb{N}$$

where $r \in \mathbb{C}$ is a constant, with $r \neq 0$. For $f : \mathbb{N}_0 \longrightarrow \mathbb{C}$, let $Af : \mathbb{N}_0 \longrightarrow \mathbb{C}$ be $(Af)(n) = f(n+1)$. We can rewrite the above as

$$(A^2 - 2rA + r^2)f = 0$$

i.e. $$(A - r)^2 f = 0.$$

One solution we can get from this computation is

$$a_n = p\,r^n, \quad n \in \mathbb{N}_0$$

where p is an arbitrary (complex) constant. If the initial conditions are $a_0 = 1$ and $a_1 = 3r$, then $a_n = p r^n, n \in \mathbb{N}_0$ does not satisfy these initial conditions for any $p \in \mathbb{C}$, for then $p = 1$ and $pr = 3r$, i.e. $p = 1$ and $p = 3$, which is not possible.

Consider $f : \mathbb{N}_0 \longrightarrow \mathbb{C}$ given by

$$f(n) = n r^n, \quad n \in \mathbb{N}_0.$$

Then

$$f(n+2) - 2r\,f(n+1) + r^2 f(n)$$
$$= (n+2)\,r^{n+2} - 2r\,(n+1)\,r^{n+1} + r^2\,(n)\,r^n$$

$$= r^{n+2}\left((n+2) - 2(n+1) + n\right) = 0.$$

This gives us a second solution. It follows that

$$g(n) = p r^n + q n r^n, \quad n \in \mathbb{N}_0$$

is a solution of $a_{n+2} - 2r a_{n+1} + r^2 a_n = 0$

for every choice of $p, q \in \mathbb{C}$. If we add initial conditions

$$a_0 = \alpha, \quad a_1 = \beta$$

then we wish to choose $p, q \in \mathbb{C}$ s.t.

$$g(0) = \alpha \text{ and } g(1) = \beta.$$

Can we find such p and q? Now p and q must satisfy

$$p = \alpha \quad \text{and} \quad pr + qr = \beta$$

i.e. $$p = \alpha \quad \text{and} \quad q = \frac{\beta - \alpha r}{r}.$$

Thus, every solution can be written in the form

$$p r^n + q n r^n, \quad n \in \mathbb{N}_0$$

and if the initial conditions are $a_0 = \alpha$, $a_1 = \beta$, then the solution is

$$a_n = \alpha r^n + \frac{\beta - \alpha r}{r} n r^n, \quad n \in \mathbb{N}_0.$$

As we argued earlier this is the unique solution of the recurrence relation which satisfies $a_0 = \alpha$ and $a_1 = \beta$.

Remark: More generally, with r as above and $m \in \mathbb{N}$, and $g_1, \ldots, g_m$ the functions $g_i(n) = n^{i-1} r^n$, the recurrence relation

$$(A - r)^m f = 0$$

has $g_1, \ldots, g_m$ as solutions and every solution is of the form $p_1 g_1 + p_2 g_2 + \ldots + p_m g_m$, where $p_1, \ldots, p_m$ are arbitrary complex numbers. In other words, every solution looks like

$$p_1 r^n + p_2 n r^n + p_3 n^2 r^n + \ldots + p_m n^{m-1} r^n, \quad n \in \mathbb{N}_0.$$

We will not prove this, but here are the main steps in the argument. Suppose $a_0 = \alpha_0, a_1 = \alpha_1, \ldots, a_{m-1} = \alpha_{m-1}$ are the

initial conditions for our recurrence relation
$$\sum_{k=0}^{m} \binom{m}{k} r^{m-k} a_{n+k} = 0 \quad , \quad n \in \mathbb{N}_0 .$$

(This is the same as $(A-r)^m f = 0$.) Then the equations
$\sum_{j=1}^{m} i^{j-1} r^i p_j$ $\quad \sum_{j=1}^{m} i^{j-1} r^i p_j = \alpha_i \, , \quad i = 0, \dots, m-1 \quad (0^0 = 1 \text{ here})$
form a system of m linear equations in the m uknowns $p_1, \dots, p_m$. It can be re-written as
$$\begin{pmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & & \\ a_{1m} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} p_1 \\ \vdots \\ p_m \end{pmatrix} = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{m-1} \end{pmatrix}$$
with $a_{ij} = (i-1)^{j-1} r^{i-1}$, $i = 1, \dots, m$, $j = 1, \dots, m$ (with $0^0 = 1$). One shows (we won't), that the coefficient matrix is invertible and hence the system has a unique solution for $p_1, \dots, p_m$. //

The general theory

Let $c_0, c_1, c_2, \dots, c_k \in \mathbb{C}$, with $c_0 \neq 0$ and $c_k \neq 0$. Let $g : \mathbb{N}_0 \longrightarrow \mathbb{C}$ be a function. A recurrence relation of the form:

$$c_0 a_{n+k} + c_1 a_{n+k-1} + \dots + c_{k-1} a_{n+1} + c_k a_n = g \quad , \quad n \in \mathbb{N} \qquad (*)$$

is called a <u>linear recurrence relation</u>.

If we know k succesive values of the a_n, say $a_r, a_{r+1}, \dots, a_{r+k-1}$, then clearly one can work out a_{r+k} from the above equation (since $c_0 \neq 0$), and in fact a_{r+n} for all $n \geq k$. In particular if we know $a_0, a_1, \dots, a_{k-1}$, then we know a_n for all $n \in \mathbb{N}_0$.

Another way of phrasing what we just said is:
The linear recurrence relation
$$c_0 a_{n+k} + c_1 a_{n+k-1} + c_2 a_{n+k-2} + \dots + c_{k-1} a_{n+1} + c_k a_n = g(n)$$
together with the initial conditions $a_0 = \alpha_0, a_1 = \alpha_1, \dots, a_{k-1} = \alpha_{k-1}$ ($\alpha_0, \dots, \alpha_{k-1} \in \mathbb{C}$ fixed) has a <u>unique solution</u>.

Remark: The set V of maps $f: \mathbb{N}_0 \longrightarrow \mathbb{C}$ is a vector space (of infinite dimension). The advancement operator $A: V \longrightarrow V$ is a linear map, and therefore so is any polynomial of the form $\alpha_0 A^k + \alpha_1 A^{k-1} + \dots + \alpha_{k-1} A + \alpha_k$, $\alpha_i \in \mathbb{C}$, $i = 0, 1, \dots, k$.

The recurrence relation (*) above can be rewritten as

$$\left(c_0 A^k + c_1 A^{k-1} + \dots + c_{k-1} A + c_k\right) f = g. \qquad (**)$$

Let Φ be the polynomial

$$\Phi(x) = \sum_{i=0}^{k} c_{k-i} x^i.$$

Then (*) (or (**)) can be written as

$$\Phi(A) f = g \qquad (**)'$$

The homogenous case: Suppose $g = 0$ in (*) (and hence in (**) and (**)'). Then (*) is called a homogenous recurrence relation. The equation then is

$$\Phi(A) f = 0 \qquad (+)$$

Recall that we have assumed that c_0 and c_k are non-zero.

Let S be the subset of V which satisfy the above homogeneous recurrence relation. Since $\Phi(A): V \longrightarrow V$ is a linear operator, therefore S is a linear subspace of V, being the null space of the linear operator $\Phi(A)$.

We claim S is k-dimensional. In fact we have a map

$$T: S \longrightarrow \mathbb{C}^k$$

given by

$$Tg = (g(0), g(1), \dots, g(k-1)).$$

This is clearly linear. Conversely, suppose

$$(\alpha_0, \dots, \alpha_{k-1}) \in \mathbb{C}^k.$$

Let g be the unique solution of our recurrence relation with initial conditions $a_0 = \alpha_0, a_1 = \alpha_1, \dots, a_{k-1} = \alpha_{k-1}$. Then $Tg = (\alpha_0, \dots, \alpha_{k-1})$. Since g is unique (given $(\alpha_0, \dots, \alpha_{k-1}) \in \mathbb{R}^k$), therefore T is bijective. Thus

$$S \cong \mathbb{C}^k.$$

It follows that S is a k-dimensional vector space. We record this as a lemma.

Lemma: The space of solutions S is a k-dimensional linear subspace of V.

The task then is to find k linearly independent solutions of the recurrence relation (with c_0, c_k non-zero)
$$\sum_{i=1}^{k} c_{k-i}\, a_{n+i} = 0, \quad n \in \mathbb{N}_0.$$
We will illustrate the method via an example. But first some terminology. The polynomial
$$\Phi(x) = c_0 x^k + c_1 x^{k-1} + \cdots + c_{k-1} x + c_k$$
is called the characteristic polynomial of the recurrence relation $\sum_{i=1}^{k} c_{k-i}\, a_{n+i} = 0$.

Example: Let us solve
$$a_{n+6} - 16 a_{n+5} + 105 a_{n+4} - 362 a_{n+3} + 692 a_{n+2} - 696 a_{n+1} + 288 a_n = 0, \quad n \in \mathbb{N}_0$$
The characteristic polynomial is
$$\Phi(x) = x^6 - 16x^5 + 105x^4 - 362x^3 + 692x^2 - 696x + 288$$
$$= (x-4)(x-3)^2(x-2)^3.$$
From an earlier remark, we know that $g_1(n) = 4^n$ is a solution of $(A-4)f = 0$; $g_2(n) = 3^n$, $g_3(n) = n3^n$ of $(A-3)^2 f = 0$; and $g_4(n) = 2^n$, $g_5(n) = n2^n$, $g_6(n) = n^2 2^n$ of $(A-2)^3 f = 0$.

It then follows that $g_1, g_2, g_3, g_4, g_5, g_6$ are solutions of
$$\Phi(A) f = 0.$$
Let S be the solution space of the above. We know S is six dimensional. It turns out that $g_1, \dots, g_6$ are linearly independent and hence a general solution is of the form $p_1 g_1 + \dots + p_6 g_6$, i.e.
$$a_n = p_1 4^n + p_2 3^n + p_3 n 3^n + p_4 2^n + p_5 n 2^n + p_6 n^2 2^n, \quad n \in \mathbb{N}_0$$
is the general solution, where $p_1, p_2, p_3, p_4, p_5, p_6 \in \mathbb{C}$ are arbitrary constants.

Let us now return to the general question. We wish to find k linearly independent solutions of the recurrence relation

$$\sum_{i=1}^{k} c_{k-i}\, a_{n+i} = 0 \quad , \quad n \in \mathbb{N}_0 .$$

Let λ be a root of the polynomial $\Phi(x) = \sum_{i=0}^{k} c_{k-i}\, x^i$. Suppose the multiplicity of λ is m. Then there is a polynomial Ψ of degree $k-m$ such that

$$\Phi(x) = \Psi(x) \cdot (x-\lambda)^m .$$

Let $g_1, \dots, g_m$ be the functions on $\mathbb{N}_0$ given by

$$g_j(n) = n^{j-1} \lambda^n , \quad n \in \mathbb{N}_0 .$$

We remarked that $(A-\lambda)^m g_j = 0$ for $j = 1, \dots, m$. It follows that

$$\Phi(A)\, g_j = \Psi(A)\,(A-\lambda)^m g_j = 0 \ , \quad j = 1, \dots, m$$

More generally, suppose $\lambda_1, \lambda_2, \dots, \lambda_d$ are the distinct roots of $\Phi(x)$, with λ_i have multiplicity m_i, $i = 1, \dots, d$. Then the following is true.

Theorem: In the above situation (with c_0 and c_k non-zero), a function $g : \mathbb{N}_0 \longrightarrow \mathbb{C}$ is a solution of the homogeneous recurrence relation

$$c_0 a_{n+k} + c_1 c_{n+k-1} + \dots + c_{k-1} a_{n+1} + c_k a_n = 0 \ , \quad n \in \mathbb{N}_0$$

if and only if g can be written uniquely as a linear combination of the functions $g_{ij} : \mathbb{N}_0 \longrightarrow \mathbb{C}$, $j = 1, \dots, m_i$, $i = 1, \dots, d$, where

$$g_{ij}(n) = n^{j-1} \lambda_i^n \quad , \quad n \in \mathbb{N}_0,\ j = 1, \dots, m_i,\ i = 1, \dots, d .$$

In other words, there exist constants $p_{ij} \in \mathbb{C}$, such that

$$g = \sum_{i=1}^{d} \sum_{j=1}^{m_i} p_{ij}\, g_{ij} .$$

Remark: We will not provide a proof this, but the strategy is clear. One has to show that the g_{ij} are solutions of our homogeneous recurrence relation, and that they are linearly independent. Since there are k of them, they will necessarily form a basis of the k-dimensional space S. To show that the g_{ij} are in S is easy (we did some special cases of this). It is linear independence that is a little more dificult. We are not going to explore that in class, but it may be interesting for you to find a proof for yourself.

Here is a special case when one can show that g_{ij}, $j=1,\dots,m_i$, $i=1,\dots,d$ are linearly independant. Suppose $m_i = 1 \quad \forall\, i \in \{1,\dots,d\}$. Then $d=k$, and the k solutions in our list are $g_1,\dots,g_k$, $g_i(n) = \lambda_i^n$, $n \in \mathbb{N}_0$. Suppose

$$p_1 g_1 + p_2 g_2 + \dots + p_k g_k = 0$$

for some scalars $p_1,\dots,p_k \in \mathbb{C}$. This means

$$\lambda_1^n p_1 + \lambda_2^n p_2 + \dots + \lambda_k^n p_k = 0 \qquad , \qquad n \in \mathbb{N}_0$$

In particular, restricting ourselves to $n=0,\dots,k-1$ we get the matrix equation:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_k \end{pmatrix} = 0$$

The coefficient matrix on the left is the Van der Monde matrix. Its determinant is well-known to be

$$\Delta = \prod_{i<j} (\lambda_i - \lambda_j).$$

Since the λ_i's are distinct, Δ is non-zero, and hence the coefficient matrix above is invertible. Thus

$$p_1 = p_2 = \dots = p_k = 0,$$

i.e., $g_1, \dots, g_k$ are linearly independent and hence form a basis of the k-dimensional vector space S.