Newton's binomial theorem

We are going to define binomial coefficients

$$\binom{s}{n}$$

for $s \in \mathbb{R}$ and $n \in \mathbb{N}_0$.

First observe that for $m, n \in \mathbb{N}_0$, with $m \geq n$, the formula $P(m,n) = m!/(m-n)!$ for the number of permutations of length n in $[m]$ gives us

(i) $P(m, 0) = 1, \quad m \in \mathbb{N}_0$

(ii) $P(m, n) = m\, P(m-1, n-1), \quad m, n \in \mathbb{N}.$

Formula (i) is obvious and for (ii) observe that

$$m\,P(m-1, n-1) = m \frac{(m-1)!}{[(m-1)-(n-1)]!} = \frac{m!}{(m-n)!} = P(m,n).$$

Next we extend P to a function

$$P : \mathbb{R} \times \mathbb{N}_0 \longrightarrow \mathbb{R}$$

by insisting that

1. $P(s, 0) = 1 \quad \forall\, s \in \mathbb{R}$

and

2. $P(s, n) = s\, P(s-1, n-1) \quad \forall\, s \in \mathbb{R}$ and $\forall\, n \in \mathbb{N}$.

Note this is $\mathbb{N}$ and not $\mathbb{N}_0$

Since the second variable will eventually hit zero by repeatedly applying 2., using 1., the above defines $P(s,n)$ for all $s \in \mathbb{R}$ and $n \in \mathbb{N}_0$.

The binomial coeff $\binom{s}{n}$ is then defined by the formula

$$\binom{s}{n} = \frac{P(s,n)}{n!}, \quad s \in \mathbb{R},\ n \in \mathbb{N}_0.$$

Example:

$$\binom{1/2}{3} = \frac{P(\frac{1}{2}, 3)}{3!} = \frac{1}{2}\frac{P(-\frac{1}{2}, 2)}{3!} = \frac{(\frac{1}{2})(-\frac{1}{2})P(-\frac{3}{2}, 1)}{3!}$$

$$= \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})P(-\frac{5}{2}, 0)}{3!} = \frac{1}{16}.$$

Lemma: Let $s \in \mathbb{R}$. Then

(a) $\binom{s}{n} = \frac{s}{n}\binom{s-1}{n-1}$, $n \geq 1$

(b) $\binom{s}{n+1} = \frac{s-n}{n+1}\binom{s}{n}$, $n \in \mathbb{N}_0$.

Proof: Part (a) is obvious from the identity

$$P(s,n) = s\,P(s-1,n-1), \quad n \in \mathbb{N},\ s \in \mathbb{R}$$

and our definition of binomial coefficients.

For (b), it is enough to prove that

(*) $$P(s,n+1) = (s-n)\,P(s,n), \quad n \in \mathbb{N}_0,\ s \in \mathbb{R}.$$

For $n=0$, both sides of the above equal s, and hence the assertion is true.

Suppose the relation asserted in (*) holds for <u>some</u> $n \in \mathbb{N}_0$ and all $s \in \mathbb{R}$. Then

$$\begin{aligned} P(s,n+2) &= s\,P(s-1,n+1) \\ &= s\,(s-1-n)\,P(s-1,n) \quad \text{(by induction hypothesis)} \\ &= (s-(n+1))\,P(s,n+1), \end{aligned}$$

i.e. (*) holds for $n+1$ and all $s \in \mathbb{R}$. q.e.d.

Theorem (Newton's binomial Theorem):

$$(1+x)^s = \sum_{n \geq 0} \binom{s}{n} x^n, \quad s \in \mathbb{R},\ n \in \mathbb{N}_0.$$

We won't be proving Newton's theorem.

Lemma: $\binom{-1/2}{n} = \frac{(-1)^n}{4^n}\binom{2n}{n}$, $n \in \mathbb{N}_0$.

Proof: By induction on n. When $n=0$, both sides equal 1, and this establishes the base case.

Suppose $\binom{-1/2}{n} = \frac{(-1)^n}{4^n}\binom{2n}{n}$ for some $n \in \mathbb{N}_0$.

Then we have the following chain of equalities:

$$\binom{-1/2}{n+1} = \frac{(-\frac{1}{2}-n)}{n+1}\binom{-1/2}{n}.$$

$$= \frac{-(2n+1)}{2(n+1)}\binom{-1/2}{n}$$

$$= \frac{-(2n+1)}{2(n+1)}\frac{(-1)^n}{4^n}\binom{2n}{n} \quad \text{(by induction hypothesis)}$$

$$= \frac{(-1)^{n+1}}{2(4^n)}\frac{(2n+1)}{n+1}\frac{(2n)!}{n!\,n!}$$

This combines to $(2n+1)!$

This combines to $(n+1)!$

$$= \frac{(-1)^{n+1}}{2(4^n)}\frac{(2n+2)}{(2n+2)}\cdot\frac{(2n+1)!}{n!\,(n+1)!}$$

$$= \frac{(-1)^{n+1}}{2(4^n)}\frac{(2n+2)!}{2(n+1)}\cdot\frac{1}{n!\,(n+1)!}$$

$$= \frac{(-1)^{n+1}}{4^{n+1}}\frac{(2n+2)!}{(n+1)!\,(n+1)!}$$

$$= \frac{(-1)^{n+1}}{4^{n+1}}\binom{2(n+1)}{n+1}$$

q.e.d.

Corollary: $\binom{1/2}{n} = \frac{1}{2n}\frac{(-1)^{n-1}}{4^{n-1}}\binom{2(n-1)}{n-1}, \quad n \geq 1$

This is important. For $n=0$, $\binom{1/2}{0} = 1$.

Proof: For $n \in \mathbb{N}$, we have

$$\binom{1/2}{n} = \frac{1/2}{n}\binom{1/2-1}{n-1}$$

$$= \frac{1}{2n}\binom{-1/2}{n-1}$$

$$= \frac{1}{2n}\frac{(-1)^{n-1}}{4^{n-1}}\binom{2(n-1)}{n-1} \quad (n-1 \in \mathbb{N}_0, \text{ since } n \geq 1).$$

q.e.d.

Let us calculate some generating functions using our results.

Theorem: The generating function for the number of lattice paths from $(0,0)$ to (n,n) for $n \in \mathbb{N}_0$ is

$$\frac{1}{\sqrt{1-4x}}.$$

(If using diagonal lattice paths then $(0,0)$ to $(2n,0)$)

Proof: By Newton's binomial theorem

$$(1-4x)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} (-4x)^n$$

$$= \sum_{n=0}^{\infty} \frac{\cancel{(-1)^n}}{\cancel{4^n}} \binom{2n}{n} \cancel{(-1)^n} \cancel{4^n} x^n$$

$$= \sum_{n=0}^{\infty} \binom{2n}{n} x^n$$

as required. q.e.d.

We will skip Section 8.5 on (the very interesting topic of) the partition of integers and visit the topic later if there is time. (Do read the fascinating topic by yourself in the meanwhile if you have time.)

8.6 Exponential generating functions

The exponential generating function of a sequence $(a_n)_{n=0}^{\infty}$ is

$$E(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n.$$

These are very useful when the problem involves arrangements and not just selections. We will see examples of this later in this section. The rule of thumb is that if strings are involved then one uses EGFs (EGF = exponential generating functions), otherwise one uses usual generating functions. Sometimes problems are a mixture of the two situations or are of one form disguised as the other.

Examples:

1. Let $(a_n)_{n\geq 0}$ be the sequence given by $a_n = 1$, $n \in \mathbb{N}_0$. Its EGF is

$$E(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x.$$

2. $$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{n!} x^n.$$ Therefore

e^{2x} is the EGF for the number of binary strings of length n. (Recall that binary string is a $\{0,1\}$-string and looks like $(1,0,0,1,\dots,0,1,1,0)$.)

e^{3x} is the EGF of $\{0,1,2\}$-strings.

$\vdots$

e^{nx} is the EGF of $\{0,1,\dots,n-1\}$-strings.

Multiplication of EGF

Let

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$$

$$B(x) = \sum_{n=0}^{\infty} \frac{b_n}{n!} x^n.$$

Then,

$$A(x)B(x) = \sum_{n=0}^{\infty} \sum_{i+j=n} \frac{a_i b_j}{i!\, j!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i+j=n} \frac{n!}{i!\, j!} a_i b_j x^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} x^n.$$

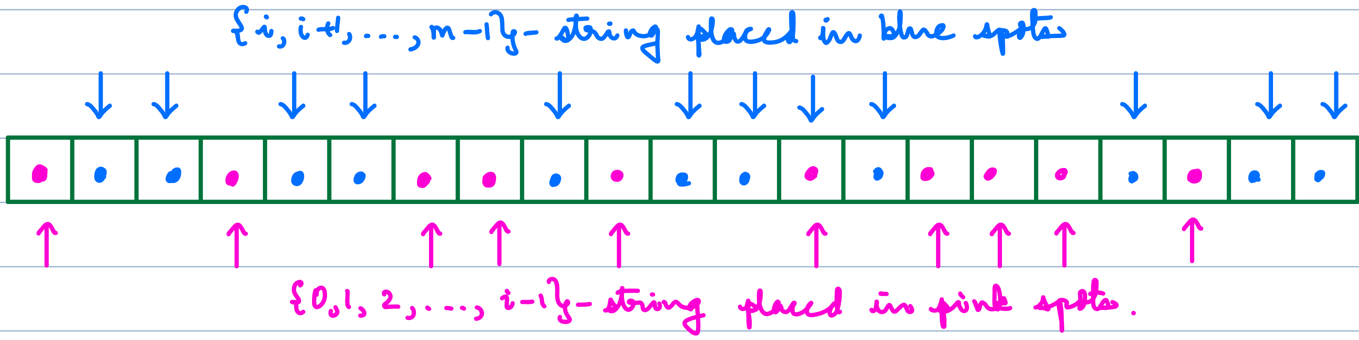
So $A(x)B(x)$ is the EGF for

$$c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.$$

Example: Suppose $i+j=m$, where $i,j \in \mathbb{N}_0$. Let $n \in \mathbb{N}_0$. Every $\{0,1,\dots,m-1\}$-string of length n can be broken up into a $\{0,1,\dots,i-1\}$-string and $\{i,i+1,\dots,m-1\}$-string.

Given a $\{0,1,\dots,i-1\}$-string of length k and a $\{i,i+1,\dots,m-1\}$ of length $n-k$, how many ways can we put them together so that we have a $\{0,1,\dots,m-1\}$-string of length n?

The trick is to pick k spots in a length n string to put the $\{0,1,\dots,i-1\}$-string in (which we have been told has length k) and use the remaining $n-k$ spots of the length n string to put the $\{i,i+1,\dots,m-1\}$-string of length $n-k$. In the picture below the pink dots represent the spots where we put the entries of the length k $\{0,1,\dots,i-1\}$-string and the blue dots represent the spots where we put the entries of the length $n-k$ $\{i,i+1,\dots,m-1\}$-string. (The order in the substrings is respected.)



Since there are $\binom{n}{k}$ possibilities for putting our $\{0,1,\dots,i-1\}$ string, and k can be any number from 0 to $m-1$,

$$\#\text{ of } m\text{-strings} = \sum_{k=0}^{n} \begin{pmatrix}\#\text{ of } i\text{-strings}\\ \text{of length } k\end{pmatrix}\begin{pmatrix}\#\text{ of } j\text{-strings}\\ \text{of length } n-k\end{pmatrix}\binom{n}{k}$$

(i-string = short form for $\{0,1,\dots,i-1\}$-string
j-string = " " " $\{0,1,\dots,j-1\}$-string
m-string = " " " $\{0,1,\dots,m-1\}$-string.)

So the EGF for $(i+j)$-strings is the product of the EGFs for i-strings and j-strings.

In other words

$$e^{(i+j)x} = e^{ix} \cdot e^{jx},$$

a formula we know well.

Example: What is the number of ternary strings of length n in which the number of zeros is even?

Solution:

Do the problem separately for strings which have only 0's in them, strings which have only 1's, and strings which only have 2's.

0: empty string, 00, 0000, ...

The EGF of ternary strings consisting of only 0's and with an even number of 0's is

$$1 + \frac{x^2}{2} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$= \frac{1}{2}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right) + \frac{1}{2}\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots\right)$$

$$= \frac{1}{2}e^{x} + \frac{1}{2}e^{-x}.$$

1: If a ternary string of length n consists of only 1, then it has an even number of 0's. Moreover there is only one ternary string of length n which consists of only 1's.

The EGF of ternary strings consisting of only 1's and with an even number of 0's $= e^x$.

2. Using the same reasoning as above we get:

The EGF of ternary strings consisting of only 2's and with an even number of 0's $= e^x$.

Therefore the required exponential generating function for our problem is

$$\frac{e^x+e^{-x}}{2}\cdot e^x\cdot e^x = \frac{1}{2}\left(e^{3x}+e^x\right)$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{3^n x^n}{n!} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{x^n}{n!}$$
$$= \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{3^n+1}{2}\right).$$

Answer: $\frac{3^n+1}{2}$.

Example: How many ternary strings of length n have at least one 0 and at least one 1?

Solution:

The idea is to find generating functions for the substrings consisting of only 0's, of only 1's, and of only 2's, and then multiplying them.

For 0's, when $n=0$ we cannot have a string which has a 0. For all other n, we have exactly one possibility, and hence the corresponding sequence of possibilities is $(0,1,1,1,\ldots)$ which gives an EGF of e^x-1.

For exactly the same reason, the EGF for 1's is e^x-1.

The EGF for the substrings consisting of only 2's is e^x.

Therefore the EGF for our problem is

$$E(x) = (e^x-1)^2 e^x$$
$$= \left(e^{2x}-2e^x+1\right)e^x$$
$$= e^{3x}-2e^{2x}+e^x$$
$$= \sum_{n=0}^{\infty}\frac{3^n-2^{n+1}+1}{n!}x^n$$

The answer is $3^n-2^{n+1}+1$.

We can check if the above answer is right by using the inclusion-exclusion principle. Let X be the set of ternary strings of length n. Let A_1 be the subset of X consisting of strings with no 0's and A_2 the subset of strings with no 1's.

Clearly $|A_1| = |A_2| = 2^n$ and $|A_1 \cap A_2| = 1$. Also $\left|\bigcap_{i\in\phi} A_i\right| = |X| = 3^n$.

Thus by the inclusion-exclusion formula we get

$$\left|X \setminus (A_1 \cup A_2)\right| = \sum_{S\subset\{1,2\}} (-1)^{|S|} \left|\bigcap_{i\in S} A_i\right|$$

$$= \left|\bigcap_{i\in\phi} A_i\right| - |A_1| - |A_2| + |A_1 \cap A_2|$$

$$= 3^n - 2^n - 2^n + 1$$

$$= 3^n - 2^{n+1} + 1$$

exactly as before.

<u>Answer:</u> $3^n - 2^{n+1} + 1$.