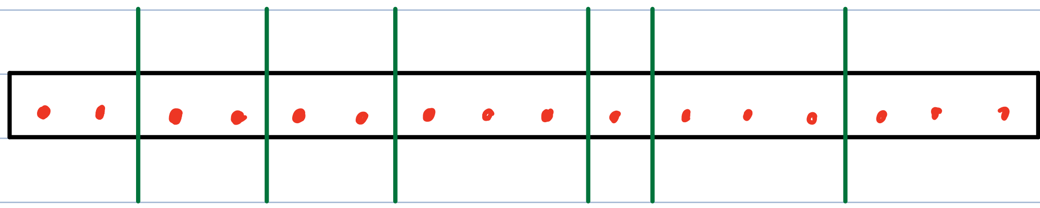
Example: How many ways can we distribute n identical apples to 7 distinct people, so that each person has at least one apple?

We saw earlier in the course that the answer is
$$\binom{n-1}{6}$$
if $n \geq 7$ (otherwise, it is obviously zero).



Seven is a large number. Let us first do the problem of distributing n identical apples to one person. The answer is a_n where

$$a_n = \begin{cases} 0 \, ; & n=0 \\ 1 \, ; & n \geq 1 \end{cases}$$

Let $F(x)$ be the generating function of $(a_n)_{n=0}^{\infty}$. Then

$$F(x) = 0 + x + x^2 + x^3 + \dots = x(1+x+x^2+\dots)$$
$$= \frac{x}{1-x}$$

Suppose c_n is the number of ways of distributing n apples amongst seven persons. Then

$$c_n = \sum_{\substack{i_1+\dots+i_7=n \\ i_j \geq 0.}} a_{i_1} a_{i_2} \dots a_{i_7}$$

← # of ways of distributing i_1 apples to the first person, i_2 to the second, ..., i_7 to the seventh.

It follows from our formula for the product of power series that the generating function $G(x) = \sum_{n=0}^{\infty} c_n x^n$ is

$$G(x) = \left(\sum_{n=0}^{\infty} a_n x^n \right)^7 = F(x)^7 = \frac{x^7}{(1-x)^7}.$$

Our formula above then gives

$$G(x) = x^7 \sum_{m=0}^{\infty} \binom{m+6}{6} x^m$$

$$= \sum_{m=0}^{\infty} \binom{m+6}{6} x^{m+7}.$$

$$= \sum_{n=7}^{\infty} \binom{n-1}{6} x^n \qquad (\text{Set } n = m+7)$$

It follows that
$$c_n = \begin{cases} 0 & n \leq 6 \\ \binom{n-1}{6} & n \geq 7. \end{cases}$$

Remark: If instead of insisting that every one of the seven persons gets at least one apple, we allowed the possibility that people can be left empty handed, then here is how we'd approach the problem: Again simplify to one person rather than seven. If a_n = # of ways of distributing n apples to one person then under the new rules, $a_0 = 1$, and $a_n = 1 \ \forall n \in \mathbb{N}_0$. The corresponding generating function is $F(x) = \frac{1}{1-x}$, and for seven persons, the corresponding generating function is

$$G(x) = \frac{1}{(1-x)^7} = \frac{1}{6!} \sum_{n=0}^{\infty} \frac{d^6}{dx^6} x^n$$

$$= \sum_{n=6}^{\infty} \binom{n}{6} x^{n-6}$$

$$= \sum_{m=0}^{\infty} \binom{m+6}{6} x^m \qquad (\text{Set } m = n-6)$$

And so the coefficient of x^n is $\binom{n+6}{6}$, and this is the # of ways to distribute n apples amongst seven people without restrictions on the minimum number of apples each person gets. Work out the details as in the example.

Example: In how many ways can we collect 25 fruits consisting of apples, oranges, pear, and bananas with the following restrictions:

- There has to be at least one banana
- Between three and seven pears
- No more than five apples.

Had "two" in lecture to section 201, but have changed it to "three"

Quick soln (with details left to you).

Generating function for bananas: $\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ $\left(\begin{smallmatrix} a_0 = 0 \\ a_n = 1, \ n \geq 1 \end{smallmatrix}\right)$

Generating function for pears: $x^3 + x^4 + x^5 + x^6 + x^7$.

$$= x^3(1 + x + x^2 + x^3 + x^4)$$

$$= x^3 \cdot \frac{1-x^5}{1-x}$$

Generating function for oranges: $\frac{1}{1-x}$ $(a_n = 1, \ n \in \mathbb{N}_0)$

Generating for apples: $1 + x + x^2 + x^3 + x^4 + x^5 = \frac{1-x^6}{1-x}$

Let $G(x) = \sum_{n=0}^{\infty} c_n x^n$ be the generating function of $(c_n)_{n \geq 0}$, where c_n is the number of ways of distributing n fruits with the above conditions. Then

$$G(x) = \frac{1-x^6}{1-x} \cdot \frac{1}{1-x} \cdot \frac{x^3(1-x^5)}{1-x} \cdot \frac{x}{1-x}$$

$$= \frac{x^4(1-x^5)(1-x^6)}{(1-x)^4}$$

$$= x^4(1 - x^5 - x^6 + x^{11}) \cdot \frac{1}{(1-x)^4}$$

$$= (x^4 - x^9 - x^{10} + x^{13}) \cdot \sum_{n=0}^{\infty} \binom{n+3}{3} x^n$$

Thus $c_{25} = \binom{24}{3} - \binom{19}{3} - \binom{18}{3} + \binom{13}{3} = 525.$

check this

Example: Find the number of solutions of

$$x_1 + x_2 + x_3 + x_4 = 45$$

with $x_1, x_2, x_3, x_4 \in \mathbb{N}_0$ and such that

$x_2 \geq 1$, $3 \leq x_3 \leq 7$, $x_5 \geq 9$. } Constraints (Had $2 \leq x_3 \leq 7$ for Section 201; but it is 3 here)

same as saying $x \in \mathbb{N}$.

The trick is to find a formula for the generating function

$$F(x) = \sum_{n=0}^{\infty} a_n x^n$$

where a_n is the number of solutions of $x_1 + x_2 + x_3 + x_4 + x_5 = n$, with the constraints indicated above.

The four generating functions whose product we have to take are (check this!)

$$F_1(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$ (gen. fcn. of # of solns of $x=n$, $x \in \mathbb{N}_0$)

$$F_2(x) = \sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$$ (gen. fcn. of # of solns of $x=n$, $x \in \mathbb{N}$)

since $x_2 \geq 1$

$$F_3(x) = x^3 + x^4 + x^5 + x^6 + x^7 = x^3(1 + x + x^2 + x^3 + x^4)$$
$$= \frac{x^3(1-x^5)}{1-x}$$

(gen. fcn. of # of solns of $x=n$, $x \in \mathbb{N}$, $3 \leq x \leq 7$)

$$F_4(x) = \sum_{n=9}^{\infty} x^n = x^9 \sum_{n=0}^{\infty} x^n = \frac{x^9}{1-x}$$

(gen. fcn of # of solns of $x=n$, $x \in \mathbb{N}$, $x \geq 9$)

We therefore have

$$F(x) = F_1(x)\, F_2(x)\, F_3(x)\, F_4(x)$$

$$= \frac{1 \cdot x \cdot x^3(1-x^5) \cdot x^9}{(1-x)^4}$$

$$= (x^{13} - x^{18}) \sum_{j=0}^{\infty} \binom{j+3}{3} x^j = \sum_{n=0}^{\infty} a_n x^n.$$

We have to find a_{45}. Clearly

$$a_{45} = \binom{35}{3} - \binom{30}{3} \quad //$$