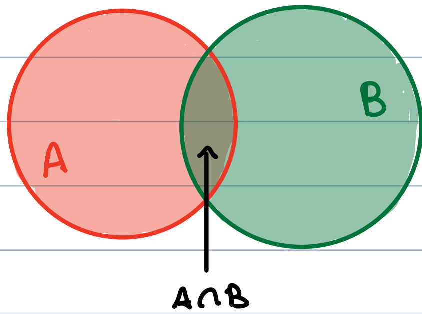
The Inclusion-Exclusion formula

Suppose A and B are finite sets.



It is clear that

$$|A \cup B| = |A| + |B| - |A \cap B|. \quad \text{— ①}$$

Indeed $|A| + |B|$ counts the elements in $A \cap B$ twice, and to compensate one needs to subtract $|A \cap B|$.

Now suppose C is another finite set. Then

$$|A \cup B \cup C| = |(A \cup B) \cup C|$$
$$= |A \cup B| + |C| - |(A \cup B) \cap C| \quad \text{(by ①).}$$
$$= \big(|A| + |B| - |A \cap B|\big) + |C| - |(A \cup B) \cap C|$$
(again by ①).

Now $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$. Putting it together we get

$$② \left\{ \begin{aligned} |(A \cup B) \cap C| &= |(A \cap C) \cup (B \cap C)| \\ &= |A \cap C| + |B \cap C| - |A \cap B \cap C| \quad \text{(again by ①)} \end{aligned} \right.$$

Substituting the above in the relation $|A \cup B \cup C| = (|A| + |B| - |A \cap B|) + |C| - |(A \cup B) \cap C|$, we get

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

It is not hard to see, using the above formulas that if D is yet another finite set

$$③ \left\{ \begin{aligned} |A \cup B \cup C \cup D| = {} & |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D| \\ & - |B \cap C| - |B \cap D| - |C \cap D| \\ & + |A \cap B \cap C| + |A \cap B \cap D| + |A \cap C \cap D| + |B \cap C \cap D| \\ & - |A \cap B \cap C \cap D|. \end{aligned} \right.$$

To see the above, write $A \cup B \cup C \cup D = (A \cup B \cup C) \cup D$, apply ① ② and ③ using the relation $(A \cup B \cup C) \cap D = (A \cap D) \cup (B \cap D) \cup (C \cap D)$.

The general result is

Theorem (I-E formula version I): Let $A_1, A_2, \dots, A_n$ be finite sets. Then

$$|A_1 \cup \dots \cup A_n| = \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le i_1 < \dots < i_k \le n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}|.$$

Observation: If $n=1$, the formula says $|A_1| = |A_1|$ which is a tautology. If $n=2$, the formula $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$ which is really ①. If $n=3$, the formula is $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$ which is ②. Check that if $n=4$, you get formula ③.

Here is a reformulation of I-E version I.

Version-II of I-E formula: Let $A_1, \dots, A_n$ be finite sets. Then

$$|A_1 \cup \dots \cup A_n| = \sum_{\phi \ne S \subset [n]} (-1)^{|S|+1} \left| \bigcap_{j \in S} A_j \right|.$$

It is clear that the two versions are equivalent.

The following is the most useful reformulation of the I-E formula.

Version III of the I-E formula: Let X be a finite set and $A_1, \dots, A_n$ subsets of X. Then

$$|X \setminus (A_1 \cup \dots \cup A_n)| = \sum_{S \subseteq [n]} (-1)^{|S|} \left| \bigcap_{j \in S} A_j \right|.$$

(What does $\bigcap_{j \in \phi} A_j$ mean?)

The convention is that

$$\boxed{\bigcap_{j \in \phi} A_j = X.}$$

Proof of equivalence of version II and III

Version II $\Rightarrow$ Version III:

$$
\begin{aligned}
|X \setminus (A_1 \cup \dots \cup A_n)| &= |X| - |A_1 \cup \dots \cup A_n| \\
&= |X| - \left\{ \sum_{\phi \neq S \subset [n]} (-1)^{|S|+1} \left| \bigcap_{j \in S} A_j \right| \right\} \\
&= |X| + \sum_{\phi \neq S \subset [n]} (-1)^{|S|} \left| \bigcap_{j \in S} A_j \right|.
\end{aligned}
$$

Now

$$X = \bigcap_{j \in \phi} A_j .$$

Hence we get

$$|X \setminus (A_1 \cup \dots \cup A_n)| = \sum_{S \subset [n]} (-1)^{|S|} \left| \bigcap_{j \in S} A_j \right|.$$

as required.

Version III $\Rightarrow$ Version II

Now $(A_1 \cup \dots \cup A_n) = X \setminus [X \setminus (A_1 \cup \dots \cup A_n)]$

So

$$
\begin{aligned}
|A_1 \cup \dots \cup A_n| &= |X| - |X \setminus (A_1 \cup \dots \cup A_n)| \\
&= |X| - \left\{ \left| \bigcap_{j \in \phi} A_j \right| + \sum_{\phi \neq S \subseteq [n]} (-1)^{|S|} \left| \bigcap_{j \in S} A_j \right| \right\} \\
&= |X| - \left\{ |X| + \sum_{\phi \neq S \subset [n]} (-1)^{|S|} \left| \bigcap_{j \in S} A_j \right| \right\} \\
&= - \sum_{\phi \neq S \subset [n]} (-1)^{|S|} \left| \bigcap_{j \in S} A_j \right| \\
&= \sum_{\phi \in S \subset [n]} (-1)^{|S|+1} \left| \bigcap_{j \in S} A_j \right|.
\end{aligned}
$$

Proof of the I-E formula

Since all versions are equivalent, it is enough to prove one version. Let us prove version III.

The case where $n=1$ is obvious (and in fact we observed this — see "Observation" after Version-I).

We will prove version III by induction on n. Suppose $n>1$ and we know the theorem for any $Y \sim (B_1 \cup \dots \cup B_r)$, with Y a finite set, $B_1, \dots, B_r$ subsets of Y, and $1 \le r < n$.

Since $A_1 \cup \dots \cup A_n = (A_1 \cup \dots \cup A_{n-1}) \cup A_n$, we have

$$|X \sim (A_1 \cup \dots \cup A_n)| = |X \sim ((A_1 \cup \dots \cup A_{n-1}) \cup A_n)|$$

$$= |X| - |A_1 \cup \dots \cup A_{n-1}| - |A_n| + |(A_1 \cup \dots \cup A_{n-1}) \cap A_n|$$

Set

$$B_i = A_i \cap A_n, \quad i = 1, \dots, n-1.$$

Then $B_1, \dots, B_{n-1}$ are subsets of A_n, and the above formula can be re-written as

$$|X \sim (A_1 \cup \dots \cup A_n)| = |X \sim (A_1 \cup \dots \cup A_{n-1})| - |A_n \sim (B_1 \cup \dots \cup B_{n-1})|.$$

Since I-E (Version III) is valid for $(n-1)$-fold unions (by our induction hypothesis), we get

$$|X \sim (A_1 \cup \dots \cup A_n)| = \sum_{T \subseteq [n-1]} (-1)^{|T|} \left|\bigcap_{i \in T} A_i\right| - \sum_{T \subseteq [n-1]} (-1)^{|T|} \left|\bigcap_{i \in T} B_i\right|$$

Now for $T \subseteq [n-1]$

$$\bigcap_{i \in T} B_i = \bigcap_{i \in T} (A_i \cap A_n)$$

$$= \left(\bigcap_{i \in T} A_i\right) \cap A_n$$

$$= \bigcap_{i \in T \cup \{n\}} A_i .$$

So we get

$$|X \setminus (A_1 \cup \dots \cup A_n)| = \sum_{\substack{T \subset [n] \\ n \notin T}} (-1)^{|T|} \left| \bigcap_{i \in T} A_i \right| - \sum_{\substack{S \subset [n] \\ n \in S}} (-1)^{|S|-1} \left| \bigcap_{j \in S} A_j \right|$$

$S = T \cup \{n\}$, $T \subseteq [n-1]$.

$$= \sum_{S \subset [n]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|$$

as required.

q.e.d.